AIDS TO EUCLID.

BOOKS I—IV.

CONTAINING

NOTES AND REMARKS ON THE PROPOSITIONS, ALTERNATIVE PROOFS, INTRODUCTORY NOTES AND 250 QUESTIONS ON THE FIRST FOUR BOOKS, FULL SOLUTIONS OF 700 EXERCISES, SOLUTIONS OF THE (LONDON, CAMBRIDGE, CALCUTTA, MADRAS, BOMBAY, PUNJAB, AND ALLAHABAD) UNIVERSITY QUESTIONS, &c., &c.

BY

A PROFESSOR OF MATHEMATICS

Fourth Edition

(Revised and Enlarged)

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PREFACE TO THE FIRST EDITION.

In preparing these "Aids to Euclid," the Editor has departed much from the usual groove, inasmuch as matters have been included herein, that are not to be found in ordinary publications of the kind, in a manner thoroughly new

The book consists of two parts. The most salient features of the one are —

(1) To help the students in studying the subject *logically*, the enunciations of all the propositions have been dissected into *data* and *quæsita*, and presented into a *tabular form*, under several heads

(2) Notes on propositions, consisting of *direct proofs*, *alternative proofs*, *immediate corollaries*, *converse propositions*, and many important matters in connection with them, have been appended, in order that they may be viewed in all their aspects.

(3) A set of all possible questions on Propositions, has been affixed, the answers of which are to be found in the "General Notes"

(4) Matriculation Papers of *four* Universities (*Calcutta*, *Cambridge*, *Madras*, *Bombay*), are relegated to the end of the book, to afford ample scope to the student for independent exercise Answers of the questions have been given in the body of the book.

In the other *part*, all *typical exercises* collected from various sources, have been fully worked out. Some of these deductions that students are desired to lay special stress upon, have been *numbered in thick type*

In short, every attempt has been made, to bring the exercises home into the comprehension of students of average understanding

Intimation of any error, will be thankfully received

PREFACE TO THE THIRD EDITION,

THE present (3rd) Edition, appears with various improvements —

(1) The *Explanatory Notes* on propositions, (which have been appended in the *first* and *second* editions, in order that they may be viewed in all their aspects)—have been considerably augmented and improved

(2) Some new *Questions* on propositions, have been added, —thus making 250 questions in all—the answers of which, are to be found in the General Notes.

(3) Many *New Exercises* have been inserted, thus making nearly 700 exercises in all—these have been selected with considerable care, from various sources

In proving these, the cumbrous and redundant Euclidian language has been abandoned, in favour of the modern, "abbreviated method" These are arranged *progressively*, so that the learner may be induced, it is hoped, from the first, to work out something for himself

(4) *Matriculation Papers* of Seven *Universities (London, Cambridge, Madras, Bombay, Calcutta, Punjab, Allahabad)* are relegated to the end of the book, to afford ample scope to the students for independent exerise—answers to the most of the questions, have been given in the *body* of the book and in the Appendix

The *Exercises*, together with *General Notes* on propositions, nearly amount to 900 in number

Care has, however, been taken to correct the misprints and inaccuracies of the *Second* Edition

In fact, the book has been entirely rewritten It only remains for me, to offer my thanks to the friends, who have improved this work, by their kind advice and suggestion

Intimation of any error, or any suggestion for its further improvement, will be most thankfully received

GENERAL NOTES ON BOOK I.

PROPOSITION I

1 In this Proposition the following axiom is assumed by Euclid, *viz* —"That a ⊙ whose centre is in the ○ce of another ⊙, must be partly within that ○ and partly without it, and therefore, that those ⊙s must necessarily cut or intersect each other"

1 In Prop 1 B I, if the two ○s intersect also at X, and AX, BX be joined, prove that ABX is another equilateral Δ

$AB=AX$ and $AB=BX$ (1 *Def* 11) $\therefore AX=BX$ (*Ax* 1.)
Thus $AB=AX=BX$, and Δ ABX is *equilateral* (1. *Def* 19)

N B —The four lines AC, AX, BC, BX, are each=AB and $\therefore$ *=to each other Hence ACBX is a rhombus or lozenge*

2 If AB be produced both ways to meet the two ○s again at D and E, prove that DE=sum of the *three* sides of the Δ ABC, also $DE=3\ AB$

$DE=AD+AB+BE=AC+AB+BC$ (1 *Def* 11) $=AB+AB+AB=3\ AB$ Thus we know how to find a st line=Sum of 3 sides of a Δ

The I 1 may be otherwise enunciated—

(1) To describe an equilateral Δ, such that each of its sides may be=to a given finite straight line

(2) To describe an equilateral Δ, such that each of its sides may be of a given length.

(3) Given two points in a plane, to find a third, such that the three points may be equidistant from one another

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PROPOSITION II

1 The *given point* is to be joined with *either* extremity of the given st line. (Let us call the extremity with which it is connected, the *connected extremity* of the given st line; and the line so connecting them, the *joining line*)

2. The centre of the 1st (smaller) ○ is the *connected extremity* of the given st line, and its radius=the given st line.

3 The equilateral Δ may be constructed on *either side* of the *joining line.*

4. The side of the equilateral △, which is produced to meet the ○, is that side which is opposite to the given point, and it is produced through the centre of the 1st ○ till it meets its ○ce.

5 The centre of the 2nd ○, is that *vertex* of the equilateral △, which is opposite to the joining line, and its radius is made up of that side of the equilateral △ which is opposite to the given point, and its production which is the radius of the 1st ○ So that, the radius of the 2nd (larger) ○, is the sum of one side of the equilateral △ and the radius of the 1st ○

6 The side of the equilateral △ which is produced through the given point to meet the 2nd ○, is that side which is opposite to the connected extremity of the given st line, and the production of this side is the line which solves the problem, for the sum of this line and the side of the equilateral △ is the radius of the 2nd ○, but also the sum of the given st line (which is the radius of the 1st ○) and a side of the equilateral △ is = to the radius of the 2nd ○ The side of the equilateral △ being taken away, the remainders are equal

As the given point may be joined with either extremity, there may be *two different joining lines*, and as the equilateral △ may be constructed on *either* side of each of these, there may be 4 different △s, so the st line and the point being given, there are 4 different constructions by which I 2 may be solved

The solution may be effected also by producing the side of the equilateral △ opposite the given point, not through the extremity of the st line, but through the vertex of the equilateral △

By I 2, the I 1 may be generalized, for an *isosceles* △ may be constructed on a given st line as base, and having its side = a given length The construction will remain unaltered, except that the radius of each of the ○s will be = to the length of the side of the proposed △ If this length be not > than half the base, the two ○s will not intersect, and no △ can be constructed

The I 2 may be otherwise enunciated thus—

(1) From a given point, to draw a straight line of a given length

(2) To find a point such that its distance from a given point may be = to a given straight line

In fact, I 2 has three cases—(1) That in which the given point is without the given line, (2) That in which the given point is in one of the extremities of the given straight line, (3) That in which the given point is in the given straight line, but is not in one of its extremities.

PROPOSITION III

By a similar operation, *the lesser line* could be produced = to the *greater*; thus from either extremity of the lesser line, let a line be drawn = to the greater, (1-2); then about this same extremity as a centre, describe a ⊙ with a radius = to the greater line; produce the lesser line to meet the ⊙ce of this ⊙, and it will be = to the greater line.

I. 3 is otherwise enunciated thus —In the greater of two given straight lines, to find a point such that its distance from either of the extremities may be = to the less.

PROPOSITION IV.

In the superposition of the Δs in I 4, *three* things are to be attended to —

1 The vertices of the equal ∠s are to be placed one on the other

2. Two equal sides to be placed one on the other

3 The other two equal sides are to be placed on the same side of those which are laid one upon the other.

In the demonstration of I 4, the converse of Ax 8 is assumed, namely, "Magnitudes which are equal, coincide with one another, when similarly placed"

In every Δ, there are *six magnitudes*, namely, the *three sides* and the *three angles*, and (except in *two particular cases*) when any *three* of these are given, the other *three* can be found, and the Δ determined

The following are the only *six* cases which can occur :—

1 The 3 ∠s
2. The three sides
3 Two sides and the ∠ between them.
4 Two sides and the ∠ opposite to one of them.
5. Two ∠s and the side between them
6 Two ∠s and the side opposite to one of them

The 1st *case* is one of the two, in which the Δ is not determined; for a Δ may have its sides increased or diminished to any extent, without altering the magnitude of its ∠s.

The 2nd *case* is demonstrated in I. 8
The 3rd *case* is the subject of I. 4.
The 4th *case* is the other one, in which the Δ is not determined,

for it is possible to have two Δs, having two sides of the one = to two sides of the other, and one of the opposite $\angle$s of the one = to the similar $\angle$ of the other, and yet the Δs themselves may not be equal Thus, let ABC be a Δ in which neither $\angle A$ nor $\angle C$ are rt. $\angle$s, and $\angle A$ is < than $\angle C$; then from B as centre, and the distance BC as radius, describe a ○ cutting AC in D, and draw DB Now it is evident that, in the Δs ABC and ABD, we have the two sides AB and BC = to the two sides AB and BD, and the opposite $\angle A$, the same in both, and yet the two Δs are not equal.

The I 4 may be otherwise enunciated thus—If the two sides and the contained $\angle$ of one Δ be respectively = to the sides and the contained $\angle$ of the other, the Δs are equal in every respect

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PROPOSITION V

1 I 5 may also be proved by superposition.

If the Δ ABC be turned over on the *plane* so that the position of the point A may be unaltered, while the side AB lies on AC, then since the $\angle A$ is the same in both, the side AC must fall on AB, and ∴ the side AB = side AC (Hyp), the point C will coincide with the point B, the point B with C, the $\angle$ ACB with the $\angle$ ABC, and the $\angle$ BCG with the $\angle$ CBF, and ∴ (1) the $\angle$ ACB will be = to the $\angle$ ABC (Ax 8), and (2) the $\angle$ BCG = to the $\angle$ CBF (Ax 8)

2 *The* $\angle$*s at the base of an isosceles* Δ *are equal to one another*, may be proved *without producing the sides of the* Δ, thus —

Let ABC be an isosceles Δ, of which side AB = side AC. Then the $\angle$ ABC shall be = to the $\angle$ ACB. In AB take any point D, from AC the greater, cut off AE = to AD the less (I 3) Join DC, BE and DE, $AB = AC$ (Hyp), and $AD = AE$ (Constr), the two sides BA, AE are = to the two sides CA, AD, and they contain the $\angle BAC$ common to the two Δs BAE, CAD, the base BE is = to the base CD (I. 4), and the Δ BAE is = to the Δ CAD, also the remaining $\angle$s of the one are = to the remaining $\angle$s of the other, *viz*, the $\angle AEB$ = to the $\angle ADC$, and the $\angle ABE$ = the $\angle ACD$ And the *whole* AB is = to the *whole* AC, of which the parts AD, AE are equal; the remainder DB is = to the remainder EC (Ax 3), and BE has been proved to be = to CD hence, the two sides DB, BE are = to the two sides EC, DC, each to each; and the $\angle$ DBE has been

proved to be = to the ∠ ECD, also the base DE is common to the 2 △s DBE, ECD; ∴ these △s are equal (I. 4.), and their remaining ∠s, viz. the ∠ BDE is = to the ∠ CED, and the ∠ BED = to the ∠CDE. Now if the equal ∠s BED, CDE be taken from the equal ∠s BDE, CED, the remaining ∠s BDC, CEB are = to one another. Again, BD, DC are = to CE, EB, and ∠ BDC is = to the ∠CEB, and the base BC is common to the two △s BDC, CEB; ∴ the 2 △s are equal in every respect (I. 4.); ∴ the ∠ DBC = the ∠ECB.

CONVERSE OF THE SECOND PART OF I. 5.

3. *The two sides of a △ being produced, if the ∠s on the other side of the base be equal, the △ is isosceles.*

Let ABC be a △, of which the sides AB, AC are produced to D and E respectively. The ∠DBC is = to the ∠ ECB. Then the △ ABC shall be isosceles. From B darw BH at rt. ∠s to AD, and from C draw CH at rt. ∠s to AE (I. 11.) The ∠ DBC = to the ∠ ECB, (Hyp.), and the ∠ DBH is = to the ∠ ECH, (Ax. 11.). ∴ the ∠ HBC = the ∠ HCB (Ax. 3.) Again ∵ ∠ ABH is = to ∠ACH, (Ax. 11.) and ∠ HBC is = to ∠ HCB, ∴ ∠ ABC = ∠ACB, (Ax. 3.); ∴ △ ABC is isosceles (I. 6.)

OTHERWISE

The ∠s ABC, DBC are together = to two rt. ∠s (I. 13), and ∠s ACB, ECB are together = to 2 rt. ∠s (I. 13); ∴ the ∠s ABC, DBC are together = to the ∠s ACB, ECB; but ∠ DBC is = to ∠ FCB (Hyp.); ∴ the △ ABC is an isosceles. (I. 6.)

ALTERNATIVE PROOF OF I. 5.

When the equal sides are produced through the vertex.

Produce BA, CA to J, H. In AJ take any pts. E and G; from AH cut off AD = AE and AF = AG. Join DG, DB and EC, EF. Then ∵ AF = AG and AE = AD and ∠ FAG is com. ∴ the base FE (of the △ FAE) = the base DG (of the △ GAD) (I. 4.) and ∠ AFE = ∠AGD (I. 4.) and ∠ FEA = ∠ GDA (I. 4.) Again ∵ in the △s FEC and GDB, BG = CF (For AB = AC and AG = AF ∴ $BA+AG$ or BG = $CA+AF$ or CF) and ∠ DGB = ∠EFC (proved), ∴ DB = EC and ∠ BDG = ∠ CEF (I. 4.) but ∠ GDA (part of ∠BDG) = ∠FEA (part of ∠CEF); ∴ the rem. ∠BDC = rem. ∠BEC. Again in the △s BDC and CEB, DB = EC (as proved), and DC = EB (for AB = AC and AD = AE); ∴ $DA+AC$ = or DC = $EA+AB$ or EB, and ∠BDC = ∠BEC (proved); ∴ the ∠DCB or ∠ ACB = ∠EBC or ∠ ABC. (I. 4.)

The I 5 may be otherwise enunciated thus :—The ∠s subtending the equal sides of an isosceles △, are equal ; and the exterior ∠s formed by producing the equal sides, are equal

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PROPOSITION VI

It should be observed that, *the portion = to the lesser side must be cut off from that end of* AB *next to the equal* ∠ ; otherwise, no proof can be drawn from I 4

I 6 is the converse of the first part of I 5 ; that is to say, the *hypothesis* of one, is the *predicate* of the other, and *vice versa*, which will be better seen thus —

I 5 *If two sides are equal*, the opposite ∠s are equal

I. 6 *If two ∠s are equal*, the opposite sides are equal

DIRECT PROOF OF I 6

Let *ABC* be a △, having the ∠*ABC* = ∠*ACB* Then the side *AB* shall be = to the side *AC* Bisect the ∠*BAC* by *AD* meeting *BC* at the pt *D* (I 9), ∵ ∠*ABC* = ∠*ACB* (Hyp), and the ∠ *BAD* = the ∠*CAD*, (Constr) and *AD* is common to the two △s *ABD* and *ACD*, the two △s *ABD* and *ACD* are = to one another in every respect (I 26), ∴ the side *AB* is = to the side *AC*.

OTHERWISE.

Bisect *BC* at the point *D* (I 10), join *AD* Let the △ *BAD* be *turned over upon the* △ *CAD*, so that *BD* may fall on *CD* ; ∵ *BD* is = to *CD* (Constr), the pt *B* coincides with *C*, and the pt *D* with the pt. *D*. The side *BA* shall fall on *CA*, ∵ the ∠ at *B* is = to the ∠ at *C* (Hyp), and the extremity *A* of the side *BA* shall coincide with the extremity *A* of the side *CA*, *DA* is common to both, ∴ the side *AB* is = side *AC*

The I 6 may be otherwise enunciated thus —If from a given point, two straight lines be drawn to meet a third, and make equal ∠s with it, these two straight lines shall be equal.

The demonstration of I 6 is indirect The proof of a proposition is said to be indirect, when its truth is established by proving its contrary to be false

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PROPOSITION VII

Alternative enunciation of I 7

On the same base, and on the same side of it, there cannot be two △*s, having their conterminous sides equal*

Third case—Let the vertex D of one Δ be on the side AB of the other, and it is evident that the sides AB and BD are not equal

Hence in no case, can two Δs, whose *conterminous sides* are equal, be constructed at the same side of the given st line.

The I. 7 may be otherwise enunciated thus :—If there be two Δs on the same base and on the same side of it, having the sides terminated at one extremity of the base = to each other, the sides terminated at the other extremity of the base, shall not be = to each other.

PROPOSITION VIII.

The following proof of I 8, is due to Proclus. It establishes the I. 8 by the I. 5

Let the two equal *bases* be so applied one upon the other, that the equal sides shall be *conterminous*, and that the Δs shall lie at opposite sides of them, and let a st line be drawn joining the vertices G and B

1. Let BG intersect the base

Let the vertex F fall at G, the side EF in the position AG, and DF in the position CG Since $BA=AG$, the $\angle GBA=\angle BGA$ (I 5), also $CB=CG$, $\therefore$ $\angle CGB=\angle CBA$ (I. 5) Adding these equals to the former, the $\angle ABC=\angle AGC$, that is, $\angle EFD=\angle ABC$.

2 Let the st. line GB fall *outside the coincident bases.*

The $\angle GBA=\angle BGA$, and also $\angle BGC=\angle GBC$ (proved) and taking the former, from the latter, the remainders, which are the $\angle$s AGC and ABC, are equal, but $\angle AGC$ is the $\angle F$.

3. Let the st line BG pass through *either extremity of the base*

In this case, it follows that the $\angle ABC=\angle AGC$ (I. 5) for the lines BA and AG must coincide with BG, since each has two points upon it.

Hence in every case the $\angle$s B and F are equal.

I 8 is the converse of I. 4.

When a theorem has *several hypothesis* and *one predicate*, if another theorem be framed having one of those hypothesis for its predicate, and the predicate of the first as one of its hypothesis, the two theorems are the *converse* of each other, and it is in this sense, that the I 8 is the *converse* of I. 4, as will be immediately seen by expressing them in the following manner .—

I 4 —If *two sides are equal*
and $\left\{\begin{array}{l}\textit{the } \angle \textit{ opposite} \\ \textit{the bases are equal}\end{array}\right\}$ then, the *bases* are equal

I. 8 —If *two sides are equal* and *the bases are equal,* $\Big\}$ then $\left\{\begin{array}{l}\text{the } \angle\text{s opposite} \\ \text{the bases, are equal}\end{array}\right.$

The I 8 may be otherwise enunciated thus :—If two Δs have the three sides of the one = to the three sides of the other, each to each, the three $\angle$s of the one shall be = to the three $\angle$s of the other, each to each, namely those to which the equal sides are opposite

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PROPOSITION IX

It is necessary that the Δ be constructed on a *different side of the joining line DE*, from that on which the given $\angle$ is placed, lest the vertex F of the equilateral Δ should happen to coincide with the vertex A of the given $\angle$, in which case, there would be no joining line FA, and therefore no *solution*

In these cases, however, in which the vertex of the equilateral Δ does not coincide with that of the given $\angle$, the I 9 can be solved by constructing the equilateral Δ on the *same side* of the joining line DE with the given $\angle$

Separate demonstrations are necessary for the *two positions* which the vertices may assume

Case 1 Let the vertex of the equilateral Δ DEF fall *within* the given $\angle$ BAC

From the two Δs DAF and EAF, the $\angle$ DAF may be proved = to the $\angle$ EAF (I 8) *i e* the $\angle$ BAC is bisected by AF

Case 2 Let the vertex of the given $\angle$, be *within* the equilateral Δ

The line FA produced will in this case bisect the $\angle$, for the *three* sides of the Δ DFA are respectively = to those of the Δ EFA Hence the $\angle$ $DFA = \angle$ EFA (I 8) Also in the Δs DFG and EFG, the side $DF = EF$, the side GF is common, and $\angle$ $DFG = \angle$ EFG, hence the base $DG = EG$ (I 4) and $\angle$ $DGA = \angle$ EGA Again, in the Δs DGA and EGA, the side $DG = EG$, and AG is common, and the $\angle$s at G are equal, hence $\angle DAG = \angle$ EAG (I 4), and $\therefore$ the $\angle$ BAC is bisected by AG

It is evident, that an isosceles Δ constructed on the joining line DE, would equally answer the purpose of the solution

N. B—By a repetition of I. 9. any ∠ may be divided into 4, 8, 16. &c. *equal* parts; *i e.*, into any number of *equal* parts which can be expressed by the successive *powers* of 2.

Alternative Proof of I. 9.

Let *BAC* be the given ∠. In *AC* take any pt. *D*. and produce *AB* to E making *AE*=to *AD*: join *ED* and through *A* draw *AF* ∥ to *DE* (I. 31) Then *AF* shall bisect the ∠ *BAC*. ∵ *AE* =*AD* (constr); ∴ the ∠ *AED*=the ∠ *ADE* (I. 5). And ∵ *AF* is ∥ to *DE*; ∴ ∠ *ADE*=∠ *DAF* (I. 29); ∴ ∠ *BAF*= ∠*CAF*. *i.e.* the ∠ *BAC* is bisected by *AF*.

N. B.—The trisection of any ∠, is possible by Modern Geometry. See *Casey's Euclid*

By means of I. 9. a given rectilineal ∠ may be divided into 2^n equal parts. where *n* is any positive integer.

For the *Alternative Enunciation* of I. 9. See *C. U. E. q.* 2 of 1895.

PROPOSITION X.

Alternative Proof of I. 10, without the aid of I. 9.

Let *AB* be the given straight line. In *AB*. take any pt. *E* *closer* to *B than A* With *A* as centre. and *AE* as radius describe a ⊙ *CED*: take *BF*=*AE*. With *B* as centre and *BF* as radius. describe a ⊙ *CFD* cutting the ⊙ *CED* at *C* and *D*. Join *CD* cutting *AB* at *G* *AB* is bisected at *G*. Join *AC*, *BC*. *AD*, *BD*: in the △s *ACD* and *BCD*. ∵ *AC*=*BC*, and *CD* is common. and *AD*=*BD*. ∴ ∠*ACD*=∠*BCD*. (I. 8.)

Again ∵ *AC*. *CG*=*BC*, *CG*; and ∠*ACG* or ∠*ACD*=∠*BCG* or ∠*BCD*. ∴ the base *AG*=base *BG* *i.e.*, *AB* is bisected at *G*.

N. B—By a repetition of (I. 10), a st. line may be divided into any number of *equal* parts denoted by the successive *powers* of the number 2; *i e.* into 4. 8, 16 etc. *equal* parts.

Thus by means of I. 10. a straight line may be divided into 2^n equal parts, where *n* is a positive integer.

PROPOSITION XI.

Cor. to I. 11. *Two st. lines cannot have a common segment.*

If possible, st. lines *ABC*, *ABD* have the segment *AB* common to both From *B* draw *BE* at rt. ∠s to *AB* (I. 11.) Then ∵

ABC is a st. line, $\angle CBE = \angle EBA$ (Def 7), also . ABD is a st line (*hyp*), $\angle DBE = \angle EBA$, $\therefore$ $\angle DBE = \angle CBE$ (Ax 1) the *less*=to the *greater*, *which is impossible* (Ax. 9)

1. By I 11, a $\perp$ can be drawn at the extremity of a given line, by first *producing* the line.

2. By comparing I 11 with I 9, it will be immediately seen, that the I 11 is only a particular case of I 9, for I 9 is to bisect *any* given $\angle$, and I 11 is to bisect *that particular* $\angle$ which *a st. line forms with its continuation*

I 11 may be otherwise enunciated thus —To draw a straight line making equal $\angle$s with a given straight line from a given point in the same

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PROPOSITION XII

1 In I 12, it is *assumed* that the given line AB will be cut by the ○ in *two* points This will be evident if we consider that a portion of the ○ce of the ○, lies on each side of the line AB, and that as the ○ce is a *continuous line*, it must necessarily cross the line *twice*

2 The given st line is supposed to be *unlimited* in length, because otherwise, it might so happen that the ○ described from C, might not *cut* it at all

N B—Euclid uses the term *at* rt $\angle$s when the st line is drawn from a point *in* another straight line, as in I 11, and he uses the term $\perp$ (*perpendicular*) when the st line is drawn from a point *without* another st line, as in I 12

I 12 may be otherwise enunciated thus —From a given point without a given straight line, to draw a straight line making equal $\angle$s with the same.

PROPOSITION XIII

ALTERNATIVE PROOF OF I 13

(See figure 2 p 28 Text-Book)

$$\begin{array}{l}
\angle ABD = \angle EBD + \angle EBA = \text{a rt } \angle + \angle EBA \\
\underline{\angle ABC = \angle EBC - \angle EBA = \text{a rt } \angle - \angle EBA} \\
\angle ABD + \angle ABC = \qquad\qquad 2\text{rt } \angle\text{s}
\end{array}$$

SYMBOLICAL FORM OF THE PROOF OF I 13.

(*see figure 2. p. 28, Text-Book.*)

$\angle DBA = \angle DBE + \angle EBA$
$\angle ABC = \angle ABC$ adding

$\angle DBA + \angle ABC = \angle DBE + \angle EBA + \angle ABC$, (1)
Again $\angle EBC = \angle EBA + \angle ABC$
$\angle DBE = \angle DBE$ adding

$\angle EBC + \angle DBE = \angle DBE + \angle EBA + \angle ABC$ (2)
From (1) and (2), $\angle DBA + \angle ABC = \angle EBC + \angle DBE$
$=$2rt. $\angle$s.

N B—It is necessary, in the *enunciation* of I 13, to insert the words "*forms* $\angle$s with it," to exclude the *case* in which the line *AB stands at either extremity of CD*

I 13 may be otherwise enunciated thus :—If either of two straight lines meeting at a point, be produced, the $\angle$s formed by the two straight lines, are together=to two rt $\angle$s.

PROPOSITION XIV.

I 14 is proved by the "*reductio ad absurdum*" It is necessary that the two st lines *CB* and *BD* should be on *opposite* sides of *AB*, for otherwise, they might form $\angle$ with it, together=to 2 rt $\angle$s *without being in the same continued st line*

I 14 is the converse of I. 13

I 14 may be otherwise enunciated thus :—If *three* straight lines meet at a point, so that the $\angle$s formed on opposite sides of one of them, are=to two rt $\angle$s, the two other st lines are in one and the *same* st. line

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PROPOSITION XV.

1 The converse of I 15 may be demonstrated as follows :—

If four st lines AE, BE, CE, DE, *meet in the same point* E, *and make the vertical* $\angle$s *equal*, each *alternate pair* of lines will form one *continued* straight line

For $\angle CEA = \angle BED$ (Hyp), and $\angle CEB = \angle AED$ (Hyp.); $\therefore \angle CEA + \angle CEB = \angle BED + \angle AED$ (Ax 2); and whole 4 $\angle$s=4 rt. $\angle$s (I 15, cor.), $\therefore \angle CEA + \angle CEB$=2 rt. $\angle$s, and $\therefore$ *AE and*

EB form one continued st line (I. 14). And so it may be shown that *CE* and *ED also form one continued st line*

2. To prove ∠CEB = ∠AED (2nd part of I 15)

∠*CEB* + ∠*BED* = 2rt. ∠s (I 13)

∠*AED* + ∠*BED* = 2rt ∠s (I 13)

∴ ∠*CEB* + ∠*BED* = ∠*AED* + ∠*BED* (Ax 1)

∴ ∠*CEB* = ∠*AED*

ALTERNATIVE PROOF OF I 15, BY I 4 AND I. 5

Make *EA* = *EB* = *EC* = *ED*; ∴ ∠*BCE* = ∠*CBE*

In the Δs *BCA* and *CBD*, *BC*, *CA* = *CB*, *BD*, and ∠*ABC* = ∠*DCB*. (I 5)

∴ the base *CA* = base *BD*, and ∠*BAC* = ∠*CDB* (I 4)

In Δs *CAE* and *BDE*; *AC*, *AE* = *BD*, *DE*, and containing ∠*CAE* = containing ∠*BDE*, ∴ ∠*AEC* = ∠*BED* (I 4)

PROPOSITION XVI

N B In the construction of the I 16, the following *general directions* should be observed

1 Bisect the side of the Δ, at which are the *exterior* ∠ and the ∠ to be proved *less*

2 Join the *point of bisection* with the *opposite* ∠

3 Produce the line thus formed, through the point of bisection

4 Make the produced part = to the joining line

5 Join the extremity of the produced straight line, with the vertex of the exterior ∠

SECOND PART OF PROOF OF I 16.

Const Bisect *BC* at *O* (I 10), join *AO* Produce *AO* to *M* making *OM* = *AO* Join *CM*

Proof In the two Δs *AOB* and *MOC*, *AO*, *OB* = *MO*, *OC*, and ∠*AOB* = ∠*MOC* (I 15), ∴ ∠*ABO* or ∠*ABC* = ∠*OCM* Now ∠*BCG* is > ∠*OCM*, ∴ ∠*BCG* > ∠*ABO* or ∠*ABC*, but ∠*BCG* = ∠*ACD*, ∴ ∠*ACD* > ∠*ABC*

I 16 may be otherwise enunciated thus —If a st line fall on two others, which meet, it shall make the exterior ∠, greater than either of the interior and opposite ∠s

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PROPOSITION XVII.

I. 17 may be proved without producing a side of the Δ.

Take any pt D in BC and join AD Then the ∠s at the pt. D are together > than the sum of the ∠s B and C (I. 16). But the ∠s at D are together = to 2 rt ∠s (I. 13); ∴ ∠ B + ∠ C are < 2 rt. ∠s

I 17 is the converse of (Ax. 12).

The I. 17 may be **otherwise** enunciated thus.—If a straight line fall on two others which meet, it shall make the two interior ∠s on that side of it, on which the two other lines meet, together < than 2 rt. ∠s

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PROPOSITION XVIII.

Alternative Proofs of I. 18.

1. With A as centre and the *lesser* side AC as radius, describe the ⊙ CED cutting the base BC at E. Join AE; ∵ AE = AC; ∴ ∠AEC = ∠ACE (I. 5), and ∵ ∠AEC > ∠ABE or ∠ABC; ∴ ∠ACE or ∠ACB > ∠ABC (I. 16)

2. By producing the lesser side.

Produce the *lesser* side AB to E so that AE = AC; ∴ ∠ACE = ∠AEC (I. 5); but ∠ABC > ∠AEC (I. 16);
∴ ∠ABC is > ∠ACE, ∴ ∠ABC > ∠ACB.

By bisecting the Vertical ∠.

Bisect ∠BAC by AD (I. 9) From AC cut off AE = AB. In the Δs BAD and EAD; BA, AD = EA, AD and ∠BAD = ∠EAD (cons); ∴ ∠ABD = ∠AED (I. 4). But ∠AED > ∠ACD or ∠ACB (I. 16); ∴ ∠ABD or ∠ABC > ∠ACB.

I. 18 may be **otherwise** enunciated thus —If one side of a Δ, be > than another, then the ∠ opposite to the greater side, shall be > than the ∠ opposite to the *less*

PROPOSITION XIX.

Direct Proof of I. 19.

Let ABC be a Δ, and BC be the base.

(*Cons*) Bisect BC in X, (I. 10) Join AX; produce AX to Y so that XY = AX, join BY. In the Δs BXY and AXC; BX,

$XY = CX$, XA, and the contained $\angle$s equal (I 15), $BY = AC$ and $\angle XBY = \angle XCA$ (I 4); but $\angle ABX > \angle XCA$ (Hyp) $\angle ABX > \angle YBX$ Bisect $\angle ABY$ by BM, produce BM to N, making $MN = BM$ (from above, it is evident, that BM *falls above* BX or BC); $\because$ $YX = AX$, $\therefore$ $YM > AM$ From MY cut off $MG = AM$ Join NG and produce it to meet BY in S

(*Proof*) In the Δs AMB and NMG, the sides AM, MB in one $= GM$, MN in the other, and $\angle AMB = \angle GMN$ (I 15), $\therefore$ $AB = NG$, and $\angle ABM = \angle GNM$, but $\angle ABM = \angle SBM$ (const), $\therefore$ $\angle GNM$ or $\angle SNM = \angle SBM$ $SN = SB$ (I 6), but $YB > SB$ or SN, and since $SN > GN$, $\therefore$ YB is much more $>$ than GN It is proved that $YB = AC$, and $GN = AB$; $AC > AB$

Prop I 19 is the converse of Prop I 18

Props I 5 and I 18 may be included in one enunciation thus —"One $\angle$ of a Δ, is $>$ or $<$ than another (I 18), or $=$ to it (I 5), according as the side opposite to the one is $>$ or $<$ than, or $=$ to the side opposite to the other"

Props I. 6 and I 19 also may be included in one enunciation thus —"One *side* of a Δ, is $>$ or $<$ than another (I 19), or $=$ to it (I 6), according as the $\angle$ opposite to the one, is $>$ or $<$ than, or $=$ to the $\angle$ opposite to the other"

I 19 may be **otherwise enunciated** thus —If one $\angle$ of a Δ, be $>$ than another, then the side opposite to the greater $\angle$ shall be $>$ than the side opposite to the less

The mutual relation of I 5, I 6, I 18, I 19 may be shown in the following manner —

$$\left\{\begin{array}{l}\text{I. 5—If } AB = AC,\text{ then } \angle C = \angle B\\ \text{I 6—If } \angle C = \angle B,\text{ then } AB = AC\end{array}\right.$$

$$\left\{\begin{array}{l}\text{I 18—If } AB > AC,\text{ then } \angle C > \angle B\\ \text{I 19—If } \angle C > \angle B,\text{ then } AB > AC\end{array}\right.$$

The *propositions* connected by a bracket, are *converse* of each other; because that which is the *hypothesis* in the *one*, is the *predicate* in the *other*

PROPOSITION XX.

Archimedes ridicules I 20 as being *self evident*, and contends that it should be therefore one of the *axioms* That a truth is considered self evident is, however, not a sufficient reason why it should be adopted as a geometrical axiom.

Directions.

1 Produce one of the sides of the Δ, the sum of which is to be proved > than the third side, through the point where they meet

2 Make the produced part = to the side adjacent to it

3 Join the extremity of the produced part, with the vertex of the opposite $\angle$.

Alternative Proofs of I. 20

1. I. 20 is sometimes proved by bisecting the $\angle A$

Let AE bisect $\angle BAC$ The $\angle BEA >$ than $\angle EAC$, and the $\angle CEA$ is $>$ than $\angle EAB$ (I 16), and since the parts of the $\angle A$ are equal, it follows, that each of the $\angle E$ is $>$ than each of the parts of $\angle A$, and by (I 19) it follows, that $BA >$ than BE and $AC > CE$, and $\therefore$ that the sum of the *former* $>$ the sum of the *latter*, $(BA+AC) > (BE+CE)$ or BC.

2 I 20 might also be proved by drawing a $\perp$ from $\angle A$ on the side BC

Apply (I 16) and (I 19)

"*The difference of any two sides of a* Δ, is $<$ *than the remaining side*" For $AC+BC$ are $>$ than AB (I. 20), let the side AC be taken from both, and we shall have $(AC+BC)-AC >$ than the *difference* between AB and AC, or $AB-AC$ is $<$ than BC

PROPOSITION XXI

1 If the two straight lines are not drawn from the *extremities* of the base, it is possible for them to exceed the two sides of the Δ in any *ratio* $<$ than that of 2 to 1

2 The $\angle BDC$ is not necessarily $>$ than $\angle BAC$

Case 1st. of I. 21, Symbolically. *See fig. of the Text p 37*

In the Δ BAE

$$\left.\begin{array}{r}(BA+AE) > BE\\ EC=EC\end{array}\right\}\begin{array}{l}\text{(I 20).}\\ \text{adding}\end{array}$$

(a) $(BA+AC) > (BE+EC)$

In the Δ CED

$$\left.\begin{array}{r}(EC+ED) > DC\\ BD=BD\end{array}\right\}\begin{array}{l}\text{(I. 20}\\ \text{adding}\end{array}$$

(b) $(EC+EB) > (CD+BD)$

but from (a) $(BA+AC) > (BE+EC)$ and from (b) $(EC+EB) > (CD+BD)$, $\therefore$ $(BA+AC) > (CD+BD)$

Alternative Proof of I. 21. (Part 2nd).

Join AD; and produce AD to meet BC in E.

$$\left.\begin{array}{l}\angle CDE > \angle CAD \\ \angle BDE > \angle BAD\end{array}\right\} \begin{array}{l}(\text{I } 16) \\ \text{adding}\end{array}$$

$(\angle CDE + \angle BDE) > (\angle CAD + \angle BAD)$ or $\angle BDC > \angle BAC$.

PROPOSITION XXII

1 If two ⌒s intersect again at M, and MF, MG be joined; another Δ MFG will be formed on the other side of the base FG fulfilling the given conditions

2 In I 22, Euclid assumes that the two ○s will have *at least one point of intersection i e K*

To prove the above it is only necessary to show that a part of one of the ○s will be *within* and another part *without* the other.

∵ the *sum* of the radii FK and GK is $> FG$ (I 20), ∴ a part of each ○ struck with those radii, must be *within* the other, and ∵ the sum of FG and GK is $> FK$, ∴ a portion of each ○ is *without* the other, and ∴ their ○ces must cut in some point K

3 *If the sum of two of the lines were = to the third, would the ○s meet? Prove that they would not intersect, (may touch).*

From pt D, draw DE = one of the given lines A (I 2), and from D, draw $DG = B$ (I. 2), and from E, draw $EF = C$ (I 2). From the centre D with the radius DG describe a ○, and from the centre E with the radius EF describe a ○, and from a pt K of intersection of these ○s draw KD, KE The 2nd ○ cuts DE at H Let HE be produced to meet the ○ce of the 2nd ○ in L

If A were $> (B + C)$, it is evident that the ○s would not meet, one being *wholly outside* the other, and if B were $> (A + C)$, they *would not meet*, one being wholly *within* the other

If $B + C$ were = the line A, the point H and K would *coincide*, for then $DK + KE$ would $= DE$ Also if $A + C$ were $= B$, the points K and L would *coincide*, for then DK would be $= EK = DE$, or $= LD$ In the *former* case, the ○s would *touch externally*, and in the *latter*, *internally*

4 If the *three* st lines A, B, C be *equal*, I 22 becomes equivalent to I 1, and the solution will be found to agree exactly with that of the I 1

PROPOSITION XXIII.

Alternative Proof of I 23

At a given point in a given st line, to make a rectilineal $\angle =$ *to a given rectilineal* $\angle$.

Let AB be the given st line, and A the given pt in it, and $\angle\ EFD$ the given rectilineal $\angle$

It is required, (at the given pt A in the given st. line AB), to make an $\angle$, that shall be = to the given rectilineal $\angle EFD$.

In FE, FD take any pts E, D Join DE In AB (produced if necessary), take $AH=FD$ Produce BA to G making $AG=FE$. In HB (produced if necessary), take $HK=ED$ From the centre A, at the distance AG, describe the ⊙ CGL (Post 3) From the centre H, at the distance HK, describe the ◯ KCL (Post. 3), and from C one of the pts in which the two ◯s cut one another, draw the st. lines CA, CH to the pts A, H (Post 1)

Then the $\angle HAC$ shall be = the $\angle EFD$
$\because AC=AG$, (Def 11) and $AG=FE$,
$\therefore AC=FE$, (Ax. 1)

Again $\because HC=HK$, (Def 11) and $HK=DE$,
$\therefore HC=ED$. (Ax. 1)

Now, in the two Δs CAH and DFE; AC, AH are = EF, FD; and HC is = DE, $\therefore \angle HAC$ is = $\angle EFD$ (I 8)

Hence, at a given pt A, in the given st line AB, the $\angle HAC$ is made = to the given rectilineal $\angle EFD$

PROPOSITION XXIV

Different Cases.

Without the condition that "DE is that side, which is not > than the other"

In the construction of I. 24, the condition that "*DE is to be the side which is not > than DF*" was added by Dr. Robert Simson; unless the condition be added, there will be three cases to consider.

In the Δs ABC, DEF, let $AB=DE$ and $AC=DF$, and let $\angle BAC$ be > $\angle EDF$ *Then must* BC be > EF

Apply the Δ DEF to the Δ ABC, so that DE coincides with AB Then $\because \angle EDF$ is > than $\angle BAC$, DF will fall between BA and AC, and F will fall (1) *on* or (2) *above*, or (3) *below*, BC

1 If F fall *on* BC, BF is < BC, $\therefore EF$ is < BC.

2. If F fall *above* BC, $BF+FA$ are < $BC+CA$ (I 21); and $FA=CA$, $\therefore BF$ is < BC, $\therefore EF$ is < than BC

3 If F fall *below* BC, let AF *cut* BC in O.

Then $(BO+OF)$ are $>BF$, I 20
and $(OC+AO)$ are $>AC$ I 20
$\therefore$ $(BO+OC)+(OF+OA)$ or $(BC+AF)>(BF+AC)$, and $AF=AC$, BC is $>$ than BF, and $\therefore$ EF is $>$ BC

PROPOSITION XXV

Direct Proof of I 25

On the *greater* base BC, make BG=the *lesser* base ED, and on BG construct a Δ BHG *equilateral* with EFD (I 22) Join AH cutting BC in K, and produce HG to meet AC in I

The $\angle H$ is evidently=the $\angle F$

1 Let BG be $>$ than BK

Since $BA=BH$, $\therefore$ $\angle BAH=\angle BHA$ (I 5) Also since $HG=AC$, AC or $HG>AI$ and $\therefore$ HI is $>$ than AI, and $\therefore$ $\angle HAI$ is $>\angle AHI$ (I 18) Hence if the equal $\angle$s BHA and BAH be added to these, the $\angle BAC$ will be found $>$ the $\angle BHG$, which is$=\angle F$

2 If BG be not $>$ than BK, is evident that the $\angle H$ is *less* than the $\angle A$

N B I 24 and I 25 are analogous to the I 4 and I 8, in the same manner as the I 18 and I 19 are to the I 5 and I 6 The (I 4, 8, 24, 25) might be combined together thus "If two Δs have two sides of the one, respectively = two sides of the other, the remaining side of the one will be $>$ or $<$ than, or = to the remaining side of the other, according as the $\angle$ *opposite* to it, in the one, is $>$ or $<$ or = to the $\angle$ *opposite* to it, in the other, or *vice versa*"

N B If two lines of given lengths be placed, so that, *one pair of extremities coincide*, and so that, in their *initial position* the *lesser* line is placed upon the *greater*, the distance between the other extremities, will then be the *difference* of the lines

If they be *opened* so as to form a gradually increasing *angle*, the line joining their extremities, will gradually increase, until the *angle* they include, becomes = *to two right angles*, when they will be in one continued line, and the line joining their extremities, is then *sum*

Thus the *major* and *minor* limits of this line is the *sum* and *difference* of the given lines This evidently includes the I 20

The *mutual relation* of I 4 I 8, I 24 I 25, may be shewn in the following manner —

I 4 If $AB=DE$, $AC=DF$ and $\angle A=\angle D$, then $BC=EF$.

I 8 If $AB=DE$, $AC=DF$, and $BC=EF$, then $\angle A=\angle D$

I 24 If $AB=DE$, $AC=DF$, and $\angle A>\angle D$, then $BC>EF$

I 25 If $AB=DE$, $AC=DF$, and $BC>EF$, then $\angle A>\angle D$

Here the *propositions* connected by a *bracket*, are the *converse* of each other

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PROPOSITION XXVI

1 Alternative Proof of I 26, Part I, by Superposition

If $\triangle ABC$ be applied to $\triangle DEF$, so that B may fall on E, and BC may fall along EF, then C will coincide with F, $\because BC=EF$, BA will fall along ED, $\because \angle ABC=\angle DEF$, CA will fall along FD, $\because \angle ACB=\angle DFE$

Hence, A will coincide with D, and $\triangle ABC$ with $\triangle DEF$, $\therefore AB=DE$, $AC=DF$, $\angle A=\angle D$, $\triangle ABC=DEF$

2. *To determine under what circumstances, two* $\triangle$*s having two sides equal, each to each, and the* $\angle$*s opposite to one pair of equal sides equal, shall be equal in all respects*

Let the sides AB and $BC=DE$ and EF, and the $\angle A=$ the $\angle D$ If $\angle B=\angle E$, it is evident, that the $\triangle$s are in every respect equal (I 4), and that $\angle C=\angle F$ But if $\angle B$ and $\angle E$ be *not equal*, let one $\angle B$ be $>$ the other $\angle E$, and from B, let a line BG be drawn making the $\angle ABG=\angle E$

In the $\triangle$s ABG and DEF, the $\angle A$ and $\angle ABG$ are $=\angle D$ and $\angle E$ and $AB=DE$, $\therefore$ the $\triangle$s are in every respect equal (I 26), and $BG=EF$ and $\angle BGA=\angle F$ But $\because EF=BC$, and $BG=BC$, $\therefore \angle BGC=\angle BCG$ (I 5), and $\therefore \angle C$ and $\angle BGA$ or $\angle F$ are *supplemental*

3 Hence, *if two* $\triangle$*s have two sides in the one, respectively = two sides in the other, and the* $\angle$*s opposite to one pair of equal sides equal, the* $\angle$*s opposite to the other equal sides, will be either equal or* supplemental

Def **Angles** are said to be of the **same species**, when they are both *acute*, both *obtuse*, or both *right*

Hence, it follows, that if 2 Δs have two sides respectively equal, each to each, and the $\angle$ opposite to one pair of equal sides equal, the remaining $\angle$s will be *equal*, and the Δs will be in every respect *equal*, if there be any circumstances from which it may be inferred that the $\angle$s opposite to the other pair of equal sides, are of the *same species*

For in this case, if they be *not right*, they can not be *supplemental*, and must be equal, in which case, the Δs will in every respect *equal* (I 26)

If they be *both right*, the Δs will be equal, $\because$ in that case, $\angle G$ and $\angle C$ being rt $\angle$s, BG must coincide with BC, and that the ΔBGA coincide with ΔBCA, but the $\Delta BGA = \Delta EFD$

There are *several circumstances* which may determine the $\angle$s *opposite* to the other pair of equal sides, to be of the *same species*, and which all determine the *equality* of the Δs, amongst which are the following —

1 If one of the two $\angle$s opposite to the other pair of *equal* sides, be rt $\angle$, for a rt $\angle$ is its own *supplement*

2 If the $\angle$s which are given equal, be *obtuse or right*, for then the other $\angle$s must be *acute*, and $\therefore$ of the *same species*

3 If the $\angle$s which are included by the equal sides be *both right or obtuse*, for then the remaining $\angle$s must be *both acute*

4 If the equal sides opposite to $\angle$s, which are not given equal, be $<$ than the other sides, these $\angle$s must be *both acute* (I 18)

In all these cases, it may be inferred, that the Δs are in every respect equal

N B It will appear by (I 38), that if two Δs have two sides respectively equal, and the *included* $\angle$s *supplemental*, their *areas* are *equal*

Def The **area** of a figure is the *quantity of surface* within its *perimeter*

End of Section I Book I

SECTION II.

THE THEORY OF PARALLEL ST. LINES

PROPOSITION XXVII

The *condition* that, both *the straight* lines AB and CD shall be in the *same plane* is necessary to be introduced in the enunciation of I. 27, I. 28, and I. 29, for it would be possible for two st lines, to accord with the remainder of the hypothesis, and yet not to be ∥, if they were not in the *same plane*

PROPOSITION XXVIII.

Alternative Proof of I. 28

1. If AB, CD be not ∥, let them meet towards B, D in M. Then we have the *exterior* $\angle EGM$ of $\Delta\, GMH$ = the *interior* $\angle GMH$ (Hyp.), which is impossible (I. 16); ∴ AB, CD must be ∥.

2. If AB, CD be not ∥, let them meet towards B, D in M. Then we have $\angle MGH + \angle GHM = 2$ rt. $\angle$s (Hyp.), which is impossible (I. 17); ∴ AB, CD must be ∥.

PROPOSITION XXIX.

Proof of Axiom 12th.

If a st line EF meets two st lines AB and CD, so as to make the two *internal* $\angle$s BGH and GHD on the same side together < than two rt $\angle$s, these st lines AB and CD being continually produced, shall at length meet upon that side, on which are the $\angle$s which are < than two rt $\angle$s

Proof For, if AB and CD do not meet when continually produced, they are ∥ (Def. 25), and if GL be drawn, making the $\angle LGH + \angle GHD$ = two rt $\angle$s, GL will be ∥ to CD (I. 28); *i.e.*, through the point G, the two st lines GL and AB have been drawn both ∥ to the line CD, which is impossible (*Playfair's Ax.*). ∴ the st. lines AB and CD are not ∥, but shall at length meet if continually produced.

I. 29 is the converse of (I. 27 and I. 28).

PROPOSITION XXX

Alternative Proof of I 30

Let AB, BD be each of them || to PQ, then shall AB and CD be || to one another

If AB, CD be not ||, they will meet, if produced, then two st lines AB, CD which intersect each other, will both be || to the same st line PQ, which is impossible, (*Playfair's Axiom*)
AB is || to CD

PROPOSITION XXXI

Alternative Proof of I 31

To draw a st line through a given pt || to a given st line

Let A be the pt and BC the given st line

It is required to draw, through the pt A, a st line || to BC From the pt A, draw $AD \perp$ to BC (I 12)

Again, from A draw, AE at rt $\angle$s to AD (I 11), produce EA to F Then EF shall be || to BC

The st line AD meets the two st lines EF, BC and makes the *alternate* $\angle$s EAD, ADC equal to one another, $\therefore$ EF is || to BC (I 27)

Hence, through the pt A, a st line EAF has been drawn || to BC

PROPOSITION XXXII

Alternative Proof of I 32, case 2nd

Prove that the three interior $\angle$s of every Δ, are together = 2 rt $\angle$s

I *Without producing a side of the Δ*

Let ABC be a Δ Then the 3 *interior* $\angle$s ACB, ABC, BAC shall be together = two rt $\angle$s Through the pt C, draw CD || to the side BA (I 31) Then $\because$ CD is || BA, and BC falls upon them, the 2 *interior* $\angle$s ABC and BCD are together = 2 rt $\angle$s (I 29) Again AB is || to CD, and AC meets them; $\angle ACD = \angle CAB$ (I. 29) To each of these equals, add the $\angle$s ABC, ACB, $\therefore$ the 3 $\angle$s of the Δ ABC, are = the $\angle$s ABC and BCD, but it has been proved, that the $\angle$s ABC, BCD are together = two rt $\angle$s, the 3 $\angle$s of the Δ ABC, are together = two rt $\angle$s

Pythagorean Proof.

2. *By drawing through the vertex A, a st line DF, ∥ to the base BC*

∵ DF is ∥ to BC, and AC meets them, ∴ ∠ FAC = ∠ ACB (I 29); ∴ DF is ∥ to BC, and AB meets them, ∴ ∠DAB = ∠ABC (I 29), but ∠DAB + ∠FAB = 2 rt ∠s, or ∠DAB + ∠BAC + ∠ FAC = 2 rt ∠s, but ∠ DAB + ∠ FAC = ∠ ABC + ∠ ABC. To each of these add ∠BAC, ∴ ∠DAB + ∠FAC + ∠ BAC = ∠ABC + ∠ACB + ∠BAC, but ∠DAB + ∠FAC + ∠BAC = 2rt. ∠s; ∴ 2 rt ∠s = the 3 ∠s of any Δ

3 *By joining the vertex A to any point D in the base BC*

∠ADC = ∠BAD + ∠ABD also ∠ ADB = ∠DAC + ∠ACD (I 32, *part 1*) Adding we have, ∠ADC + ∠ADB or 2 rt ∠s (I 13) = 3 ∠s of the Δ ABC

Another Proof of Simson's 1st Corollary to I 32.

A Pentagon (*five sided figure*) may be divided into 3 Δs *i e* (5—2) Δs by joining *one vertex* of the figure, to each of the other *vertices*

A Hexagon (*six-sided figure*) may be divided into 4 Δs *i e.* (6—2) Δs, and thus it is evident, that a figure of n sides may be divided into $(n-2)$ ∠s Now 3 Δs of a Δ, are together = 2 rt ∠s (I 32 *part 2nd*), ∴ sum of the *interior* ∠s of $(n-2)$ Δs = $2(n-2)$ or $(2n-4)$ rt ∠s.

Thus the *interior* ∠s of a figure with n sides = $(2n-4)$ rt ∠s; but there are **n** *interior* ∠s *in a figure of* **n** *sides.*

Thus we have by Ax 1

n interior ∠s of the figure = $(2n-4)$ rt ∠s, adding 4 rt ∠s to each, we have, n interior ∠s + 4 rt ∠s = $2n$ rt ∠s

N B From above it is evident, that, *if the figure* be *regular*, the interior ∠s are *equal*, and ∴ *one interior* ∠ of the figure (A)

$$=\frac{2n-4}{n}\text{rt } \angle\text{s} = \frac{2(n-2)}{n}\text{rt. } \angle\text{s} = \frac{2(n-2)}{n} \times 90 \textit{ degrees}$$

1 Rule for finding the value of an ∠ of a regular figure

All the interior ∠s + 4 rt ∠s = (2 × no of sides) rt ∠s.

All the interior ∠s = (2 × no of sides) rt ∠s − 4 rt ∠s

$$\text{One interior } \angle = \frac{(2 \times \text{no of sides}) \text{ rt } \angle\text{s} - 4 \text{ rt } \angle\text{s}}{\text{no of interior } \angle\text{s}}$$

2 *Rule for finding the* "number" *of sides of a regular figure of* n *sides, when the value of one interior* $\angle$ *is given* — Let x rt $\angle$s = one interior $\angle$, then from (A), $nx=2n-4$, or $4=(2-x)n$, $\therefore n=\frac{4}{2-x}$ (B)

Another Proof of Simson's 2nd Corollary I 32

From any angular pt A of a rectilineal figure, draw st lines $\parallel$ to the sides of the figure It is evident that if two st lines be $\parallel$ to two other st lines each to each, the *1st pair* make the same $\angle$s with one another as the *2nd* From the theorem mentioned above, the *exterior* $\angle$s of the figure, may be found = the $\angle$s at A which are together = 4 rt $\angle$s (I 15 Cor 2)

N B Simson's *1st corollary* applies to *all rectilineal figures*, whether convex *or not*, but the *2nd* only to convex *figures i e* figures, which have no *re-entrant* $\angle$s

---

PROPOSITION XXXIII

I 33 Can be otherwise enunciated thus —The st. lines "which without crossing each other," join the extremities of two equal and $\parallel$ lines, are themselves equal and $\parallel$

The words "*towards the same parts*" are a necessary restriction, for if they were omitted, it would be doubtful whether the extremities A and C, and B and D, were to be joined by the lines AC and BD, or the extremities A,D, and B,C, by the lines AD and BC

N B St lines, which join the ends of two equal and $\parallel$ st lines "towards the *opposite* parts," shall *bisect* each other

---

PROPOSITION XXXIV

Converse of I 34

1 *If the opposite sides of a quadrilateral figure be equal, it is a* □m

Let $ABCD$ be a quadl fig, having the side AB = the side CD, and $AD = BC$ Then the fig would be a □m, join AC, $AB = CD$ (Hyp), and AC is common to the $\triangle$s ABC, CDA, the two sides BA, AC = the two sides DC, CA, and the base

BC=the base DA (Hyp.). $\because$ the $\triangle$s are=in every respect (I 8), $\therefore \angle BAC = \angle DCA$, and they are *alternate* $\angle$s, $\therefore$ CD is $\parallel$ to BA (I 27); again $\because \angle CAD = \angle ACB$; $\therefore AD$ is $\parallel$ to BC (I 27), $\therefore$ $ABCD$ is a $\square$m

2 *If the opposite $\angle$s of a quad fig be equal, it is a $\square$m*

Let $ABCD$ be a quad fig, having the $\angle ABC$=the $\angle ADC$, and $\angle BCD = \angle BAD$ Then the figure shall be a $\square$m, now $\angle ABC + \angle BCD + \angle CDA + \angle DAB$=4 rt $\angle$s (Cor I 32), and $\because \angle DAB = \angle BCD$ (Hyp), and $\angle ABC = \angle ADC$ (Hyp), $\therefore \angle DAB + \angle ADC$=two rt $\angle$s, AB is $\parallel$ to DC (I 28), similarly, AD is $\parallel$ to BC, $\therefore$ the fig is a $\square$m

3 *If each of the diagonals bisect the quad, the fig is a $\square$m*

Let $ABCD$ be a quad fig, of which each of the diagonals AC, BD bisect the figure; then it shall be a $\square$m The $\triangle$ ADC is $\frac{1}{2}$ of the figure $ABCD$ (Hyp) and also $\triangle$ BCD is $\frac{1}{2}$ of the figure $ABCD$, (Hyp), $\therefore$ $\triangle ADC = \triangle BCD$ (Ax 7). AB is $\parallel$ to CD (I 39), similarly AD is $\parallel$ to BC, $\therefore$ the figure is a $\square$m

---

PROPOSITION XXXV.

I Symbolical form of the Proof of I 35.

Let $ABCD$, $EBCF$ be $\square$ms on the same base BC, and between the same $\parallel$s AF, BC; it is required to prove that $\square$m $ABCD$= $\square$m $EBCF$

AF meets the $\parallel$s AB, DC

interior $\angle A$=exterior $\angle FDC$ (I 29),

and $\because AF$ meets the $\parallel$s EB, FC,

$\therefore$ exterior $\angle AEB$=interior $\angle F$ (I 29)

$$\text{In } \triangle\text{s } ABE,\ DCF, \begin{cases} \angle EAB = \angle FDC \text{ (proved)} \\ \angle AEB = \angle DFC \text{ (proved)} \\ AB = DC \text{ (I 34)} \end{cases}$$

$\therefore$ $\triangle ABE = \triangle DCF$ (I 26)

Thus figure $ABCF - \triangle ABE$=figure $ABCF - \triangle DCF$, *i e* $\square$m $EBCF$=$\square$m $ABCD$

2 Converse of I 35

Equal $\square$ms on the same base and on the same side of it, are between the same $\parallel$s.

Let $ABCD$ and $EBCF$ be two equal $\square$ms on the same base BC, and on the same side of it If AD or AD produced be *not* in

the same st line with EF, let it cut BE and CF or these st. lines produced at G and H

Then $GBCH$ is a □m, and $=ABCD$ (I 35)

But fig $ABCD=$ fig $EBCF$

$GBCH=EBCF$, which is *impossible* (Ax 9) Hence AD and EF must be in the same st line, *i e* $ABCD$ and $EBCF$ are between the same ||s

---

PROPOSITION XXXVI

Alternative Proof of I 36

Produce BA, FE to meet in M Through M, draw ML || to AH or BG Produce CD to meet ML at L Join LH Now $ML=CB$ (I 34), but $BC=FG$ (Hyp)$=EH$, $ML=BH$, and ML is || to EH (Const), hence $MEHL$ is a □m ME or MF is || to LH, but EF or MF is || to HG (for $EFGH$ is a □m), hence LH and HG are in *one st line*, ∴ $MFGL$ is a □m, and $MFGL=MBCL$ (I 35), also $MEHL=MADL$ (I 35), remainder $EFGH=$ remainder $ABCD$

N B I 36 can also be proved by joining AF and DG instead of joining BE, CH

PROPOSITION XXXVII

By I 37, *we can convert any rectilineal figure into an equivalent* △

Let $ABCD$ be any *rectilineal figure*, it is required to convert it into an equivalent △ join AC, AD through B draw BF || to AC, through E draw EG || to AD (I 31), and let them meet CD produced in F and G, join AF and AG Then AFG is the △ required

For △ $AFC=$ △ ABC, and △ $AGD=$ △ AED (I 37)

∴ △ $AFC+$ △ $AGD+$ △ $ACD=$ △ $ABC+$ △ $AED+$ △ ACD

△ $AFG=$ figure $ABCDE$

PROPOSITION XXXVIII

Here the *bases* of two Δs are in the *same* st line

From I 38, it is evident, that each median (the line joining the vertex, to the middle point of the opposite side) bisects the Δ

In the figure of I 38, if the point C coincides with E, and D with A, then the $\angle$ of one, is *supplemental* to that of the other. Hence, we have "If two Δs have two sides of the one, respectively = to two sides of the other, and the contained $\angle$ *supplemental*, the two Δs are *equal in area* '

PROPOSITIONS XXXIX and XL

If the vertices of all the equal Δs which can be described on the *same* base (as in I 39) or on the *equal* bases (as in I 40) be joined, the line thus formed will be a st line and is the locus of the vertices of equal Δs on the *same* base or on *equal* bases

For *def* of *locus*, (see p 115 Text)

I 39 is the converse of I 37, and I. 40 is the converse of I. 38

I 39 and I 40 may be included in one enunciation thus:—

"*Equal* Δ*s on the same or on equal bases, in the same st line, and on the same side of it, are between the same* $\parallel$ *s* '

Direct Proof of I 40

Let the equal Δs ABC, DEF be on *equal* bases BC, EF, in the same st line BF, and towards the same part Then *they shall be between the same* $\parallel$ s. Join BD, CD The Δs BDC and EFD are equal to one another (I 38), but the ΔABC is = the ΔDEF; $\therefore$ the ΔABC is = the ΔDBC. But *equal* Δs upon the same base and on the same side of it, are between the same $\parallel$ s (I. 39), $\therefore$ AD is $\parallel$ to BC *i e.* to BF

PROPOSITION XLI

1 Alternative Proof of I 41

Through C draw a st line $CF \parallel$ to BE, meeting AE produced at F Now $\square$m $ABCD = \square$m $EBCF$ (I 35). But $\square$m $EBCF = 2\Delta EBC$ (I. 34), $\therefore$ $\square$m $ABCD = 2\Delta EBC$.

The following is the converse of I 41 —"If a □m is 2ce that of a Δ, and they have the same base, and are towards the same parts, they shall be between the same ∥ s"
Apply similar proof to I 39

3 *If a □m and a Δ, be on equal bases, and between the same ∥ s, the □ m shall be double of the Δ*

Let □m $ABCD$ and ΔEFG be on *equal* bases BC, FG, and between the same ∥ s AE, BG, join AC, $\Delta ABC = \Delta EFG$ (I 38), □m $ABCD = 2\Delta ABC$ (I 34), □m $ABCD = 2\Delta EFG$

I 41 may be generalized thus —(1) If a □m and a Δ have *equal bases and altitudes*, the □m is *double* of the Δ

(2) If a □m and a Δ, have equal *altitudes*, and the base of the Δ be *double* the base of the □m, the □m and the Δ will be equal

(3) If a □m and a Δ, have equal bases, and the altitude of the Δ be *double* the altitude of the □m, they will be equal

---

PROPOSITION XLII

1 *Describe a Δ that shall be = to a given □m, and have one of its ∠s = to a given ∠*

This is analogous to I 42, and is proved thus —

Let $FECG$ be a □m (fig I 42, Text), and D the given ∠ Produce CE to B, so that $EB = CE$ (I 3), at B, make $\angle CBA = \angle D$ (I 23), and let BA meet GF or GF produced at A, join AC Then ABC is the Δ required

---

PROPOSITION XLIII

It is evident that, the *eight* □ms, *viz* $AEKH$, $KGCF$, $EBGK$, $HKFD$, $AEFD$, $ABGH$, $HGCD$ and $EBCF$ are all *equiangular* to the □m $ABCD$

N B The *maximum value* which each *complement* can have, is *one-fourth* of the □m

---

PROPOSITION XLIV

The foot-note on p 77, Text Book, is very important

Another enunciation of I 44 —

On a given base, describe a □m, which shall be = to a given Δ, and have one of its ∠s = to a given ∠

The *construction* would have been more in *Euclid's manner*, if he had made $GH=BA$, and then joining HA, had proved that HA, was ∥ to GB, by (I 33)

PROPOSITION XLV.

In the Text Book, this *problem* (I 45) is solved only for a *rectilineal fig* of *four* sides. If the fig have *more* than *four* sides, it may be divided into Δs, by drawing str lines from any ∠ of the fig, to the opposite ∠s, and then a □m = the *third* Δ, can be *applied* to LM, and having an ∠ = ∠E and so on, for all the Δs of which the rectilineal figure is composed

By means of I 45 and I. 44, a □m can be constructed *on a given line* = in area to a given rectilineal fig, and having an ∠ = a given rectilineal ∠, by constructing on the given line, a □m = in area to the first Δ ABD

PROPOSITION XLVI

Alternative Proof of I. 46.

At A, draw AD ⊥ to AB and $=AB$; with D and B as *centres*, and a radius $=DA$ or BA, describe two ⊙s intersecting at E, join EB and ED Then $ABED$ is a *square* It is evident that $ABED$ is a *rhombus* (const), and at the same time a □m, and ∠A = a rt ∠; ∴ all the ∠s are rt ∠s, ∴ it is a *square*

2 *If two squares are constructed on equal st lines AB and CD, they are equal*

Draw the diagonals EB and GD, ∵ in the Δs EAB and GCD, the sides EA and AB are respectively $=GC$ and CD (Hyp and Def), and ∠A = ∠C, ∴ the Δs are equal (I 4) And ∵ the squares AF and CH are *doubles* of the Δs EAB and GCD (I 34), *they are equal* (Ax. 6)

3 *If two squares AF and CH are equal, their sides are equal*

For, if it be possible, let one of them AF be the *greater*, take $AK=CD$, and $AI=CG$ (I 3), and join IK Then the Δ IAK = ΔGCD (I 4); but the ΔEAB = ΔGCD, being *halves* of the equal squares AF and CH (Ax 7), ∴ the Δ IAK = Δ EAB (Ax. 1), a *part* = the *whole*, which is absurd, ∴ neither of the sides AB or CD is greater than the other, but they are equal

PROPOSITION XLVII—(*Very Important*)

I 47 is *included* as a case of the following *more general one*, taken from the *mathematical collection* of Pappus, an eminent Greek Geometer of the *4th century*

In any ΔABC, ▭*ms* AE *and* CG *being described on the sides, and their sides* DE *and* IG *being produced to meet at* H, *and* HBI *being drawn, the* ▭*m on* AC, (*whose sides are equal and* ∥ *to* BH) *is* $=AE+CG$

For, draw AK and CL ∥ to BH, to meet DH and FH in K and L Since AH is a ▭m, $AK=BH$, and for a similar reason, $CI=BH$ Hence CL and AK are = and ∥, and AL is a ▭m (I 33) The ▭m $AE=$▭m AH being on the same base AB, and between the same ∥ s and ▭m $AH=$▭m KI whose common base is AK Hence the ▭m $AE=$▭m KI So the ▭m $CG=$ ▭m LI, and ∴ $AE+CG=AL$

I 47 is a particular case of the following more general one

In any Δ ABC, *squares being constructed on the sides* AB *and* BC, *and on the base, and* ⊥*s* ADF *and* CEG *being drawn from the extremities of the base to the sides, the* ▭*m* AG *and* ▭*m* CF *formed by the segments* CD, AE, *with the sides of the squares, will be together = the square on the base* AC.

For, draw AH and BI; and also BK ⊥ to AC The ▭m KC =▭m CF So ▭m $CE=$▭CL from the proof of (I 47) So, ▭m $AK=$▭m AG, ∴ the sq on $AC=$▭ms $(AG+CF)$

(1) If the Δ be *rt* ∠*d* at B, the lines GE and DF will coincide with the sides of the squares, and the proposition will become (I 47) (2) If ∠B be *acute*, the ⊥s AD and CE will fall *within* the Δ, and the ▭ms AG and CF are < than the squares on the sides (3) If ∠B be *obtuse*, the ⊥, fall *outside* the Δ, and the ▭ms AG and CF are > than the squares on the sides

Hence I 47 may be extended thus —

The square on the base of a Δ, *is < than, = to, or > than, the sum of the squares on the sides, according as the vertical* ∠ *is < than, = to, or > than a rt* ∠

Alternative Proofs of I 47

1 Let ABC be a rt ∠d Δ, having the rt ∠BAC Then the square described on the side BC, shall be = the squares on BA, AC

On BC describe the square $BDEC$ (I 46), and on BA, AC the squares GB, HC (I 46), through A draw AL ∥ to BD or CE (I 31), produce DB to meet FG at M, and LA to meet FG

produced, at N. Now $\angle MBC = \angle ABF$, each being a rt $\angle$, take away from each, $\angle ABM$; $\therefore$ $\angle FBM = \angle ABC$ (Ax. 3). Now, $\angle ABC = \angle MBF$, and $\angle BFM = \angle BAC$ (Ax 11), and the side BF=the side BA (Def), the side MB is=the side BC (I 26) but $BC=BD$, $\therefore$ $MB=BD$ The square BG=the □m BN (I 35), but the □m BN=rect. BL (I 36), $\because$ the square BG is =□m BL (Ax 1).

So, it may be shown, that square CH is =the □m CL, $\therefore$the square BE is =squares BG. CH

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2 Join AM The Δ AMB = *half* of the square AF (I 41), and it is also *half* of BL (I 41). $\therefore$ the square FA =BL So, it may be proved, that the square AK = CL, $\therefore$ the whole square BE=squares FA, AK

---

3 Let ABC be a rt $\angle$d Δ, having the rt $\angle ABC$ Then the square described on the side AC, shall be=squares on BA, BC From A draw AG at rt $\angle$s to AC (I 11), make $AG=AC$, produce BA to D, and from G, draw $GD \perp$ to AD (I 12) So draw CH at rt $\angle$s and=CA Produce BC to F, and draw $HF \perp$ to CF (I 12) Produce DG, FH to meet in E, join GH. Through C, draw $CNK \parallel$ to BD (I 31), and through A, draw $ANL \parallel$ to BF (I 31)

Now $\because$ CB is $\parallel$ to NA, and AC meets them, $\angle CAN = \angle ACB$ (I 29) The $\angle CAG = \angle NAD$ (Ax. 11), take away from each $\angle NAG$, $\therefore$ $\angle CAN = \angle GAD$, but $\angle CAN$ has been proved to be= $\angle ACB$, $\therefore$ $\angle ACB = \angle GAD$ (Ax 10) Now, $\because$ $\angle$s GAD GDA are= $\angle$s ACB, ABC, and the side AC = the side AG (const), $\because$ the Δ $GDA = \Delta ABC$ in every respect (I 26), $\therefore$ the side DG=side BA, and $DA=BC$

So, it may be proved that, $FC=BA$, and $HF=BC$, $\therefore$ FC $=BA=DG$, and also $DA=BC=FH$, $\because$ $GA=AC$ and HC= AC. $\therefore$ $HC=GA$ (Ax. 1); and $\because$ the $\angle$s GAC, ACH are rt $\angle$s. $\therefore$ AH is a square, $GH=AC$, and the $\angle$s, GEH, EGH, are= to the $\angle$s CBA, BCA, $\therefore$ Δ $GEH = \Delta$ CBA (I 26) So it may be proved that, Δ $GDA = \Delta HFC$ The square AK=square on BC for $BC=AD$ or AN

Similarly, CL=the square on AB The □m AC is *double* of the Δ ACB (I 34), $\therefore$ KL, AC are together=*four times* the Δ ABC

Again, $\because$ Δ CAB = Δ HFC= Δ GEH = Δ ADG, these are together=*four times* the Δ ABC, $\therefore$ these four Δs are together=the □ms KL and AC.

From the figure EB, take away the □ms KL, AC, and from the figure EB, take away the *four equal* △s, and the remainders are equal (Ax 3), ∴ the square on AC=squares on AB, BC

---

4 Join GN Through N, draw PNR ∥ to HC or GA (I 31), meeting HG in P, and AC in R The △ GNA=*half* of the square KA (I 41), and it is also=*half* of the rect. $PGAR$ (I 41), ∴ the square KA=the rect. $PGAR$ (Ax 6)

So, by joining HN, it can be proved, that △ HNC=*half* of the square LC (I 41), and it is also=*half* of $PRCH$ (I 41), ∴ the rect $PRCH$=the square LC, but the square KA has been proved to be=PA, ∴ the square GC is=the squares KA, LC; but the square KA=the square on CB, and the square LC= the square on AB, ∴ the square on AC=the squares on AB, BC

Different Cases of I. 47.

1 *The greater and one of the smaller squares on the exterior sides, and the other on the interior side.*

Let ABC be a rt ∠d △, having the rt ∠BAC

Then the square on the side BC, shall be=the squares on BA, AC.

On BC describe the square $BDEC$, and on BA, AC the square GB, HC (I 46), through A, draw AL ∥ to BD or CE (I 31), and join BK, AE

Then ∠ BAC is a rt ∠ (hyp), and ∠ HAC is a rt ∠, the two st lines BA, AH on the *opposite* sides of AC, make with *it* at the point A, the adjacent ∠ s=2rt ∠s, ∴ *BA is in the same st line with AH* (I 14)

The ∠ ABF=∠ CBD, take away the common ∠CBF, the remaining ∠ ABC= ∠ FBD The two sides AB, BC in the △ ABC, are respectively=the sides KB, BD in the △ FBD, and ∠ ABC=∠ FBD, ∴ the two △s are equal to one another in every respect (I 4), ∴ ∠ BFD=∠ BAC, but ∠ BAC is a rt ∠, ∴ the ∠ BFD is also a rt ∠

∵ ∠s BFG, BFD are together=2rt ∠s, ∴ *DF is in the same st line with FG* (I 14) The square BG, and the □m AD are on the same base AB, and between the same ∥ s AB, GD, ∴ square BG= □m AD (I 35) For the same reason, AD=BL, ∴ BG= BL The △ KCB = △ ACE The △ KCB=half of the square AK (I 41), and the rect. CL is *double* of the △ ACE *Doubles of equal things are equal to one another*; ∴ AK=CL

But BL has been proved to be equilateral, ∴ the squares BG and AK are together = the square BE

2. The three squares on the interior sides.

Let ABC be a rt. ∠d △, having the rt. ∠ BAC. Then the square on the side BC, shall be = the squares on BA, AC.

On BC, describe the square $BDEC$, and on BA, AC describe the squares GB, HA (I. 46), through A, draw AL ∥ l to BD or CE (I. 31), and join AD, AE, BH, FC.

The ∠ ECB = ∠ ACH, each of them being a rt. ∠, take away the common ∠ ACB, ∴ the remaining ∠ ECA = the ∠ BCH. Also ∵ the two sides EC, CA are = the sides BC, CH, and ∠ ECA = ∠ BCH, ∴ △ ECA = △ BCH (I. 4). So, it can be shown that, △ DBA = △ FBC.

The △ ECA = ½ of LC (I. 41), and the △ BCH = ½ of KC (I. 41), ∴ LC = KC. Again, the △ DBA is = ½ of LB, and the △ FBC is = ½ of BG, ∴ the square DC is = squares BG and HA.

3. The greater one and one of the smaller squares on the interior sides, and the other on the exterior side.

Let ABC be a rt. ∠d △, having the rt. ∠ BAC. Then the square on the side BC, shall be = the squares on BA, AC.

On BC, describe the square $BDEC$, and on BA, AC the squares GB, HC (I. 46), through A, draw AL ∥ l to BD or CE (I. 31), and join CF, AE, EH, DA. Now HK is in the *same st. line with EH*. For, if HK be *not* in the same st. line with HE, produce KH to M, and CE to M.

The ∠ACB is = ∠KCM, and ∠ BAC is = ∠ CKM, each of them being a rt. ∠, and AC=KC, each being the side of a square, ∴ the side CM is = the side CB (I. 26), but CB is = CE, ∴ the side CE = the side CM, *the less side = to the greater*, which is *impossible*, ∴ HM is *not* in the same st. line with HK. And so, it may be shown that, no other st. line but HE, is in the *same st. line* with HK. The △ACE is = ½ of the rect. LC (I. 41), and *it* is also = ½ of the square HC (I. 41), ∴ the rect. LC is = the square AK (Ax. 6), but it has been shown in the *foregoing case*, that the rect. BL is = the square BG, ∴ the *whole* square DC is = *the squares* BG and AK.

4. The two smaller squares on the exterior sides, and the greater on the interior side.

Let ABC be a rt. ∠d △, having the rt. ∠BAC. The square DC on BC, shall be = to the square FA, AK described on BA, AC. Through A, draw LAM ∥l to BD or CE (I. 31), and join DA, EH. The △ABD is = *half* the rect. DM (I. 41), it is also = *half* the

square FA, and it has been proved that, the rectangle LC is = to the square AK, ∴ the square DC is = to the square FC, AK

N B In case 4th, it is evident that the angular pt D of the square DC shall be in FG From B, draw BD at rt ∠ s to BC (I 11), meeting FG at the point D Through D, draw DE ∥ l and = to BC, join FC Then DC is a square The ∠ FBD = the ∠ ABC, and ∠ BFD = ∠ BAC, and FB = BA, ∴ BD = BC (I 26) But DE = BC (Constr), ∴ DE = DB Again, ∵ BC is both = and ∥ to DE, ∴ the figure DC is an equilateral ▭, and ∴ the ∠ CBD is a rt ∠, the figure DC is rt ∠ d, but it has been proved to be equilateral ▭, ∴ it is a square

5 *The two smaller squares on the interior sides, and the greater on the exterior side*

Let ABC be a rt ∠d Δ, having the rt ∠BAC Then the square BE on BC, shall be = to the squares HB, FC described on AC, AB Through A, draw AL ∥ l to BD or CE, join KE, DF It has been proved in (I), that DF is in the same st line with FG. So, it can be proved that, HK is in the same st line with KE The ▭ AE is = to the rect CL (I 35), the ▭ AE is also = to the square CH (I 35), ∴ the square CH is = to the rectangle CL (Ax 1), but it has been proved in (I), that the square BG is = to the rectangle BL, ∴ the whole square BE = to the squares HC, BC

Rule for finding out the side of a rt ∠d Δ, when the two other sides are given

Let a be the *hypotenuse* of a rt ∠d Δ, b and c the other sides

$$a^2=b^2+c^2 \quad (\text{I } 47)$$
$$\therefore b^2=a^2-c^2$$
$$\therefore b=\sqrt{(a^2-c^2)}$$
$$\text{So, } c=\sqrt{(a^2-b^2)}$$

Rule of Pythagoras

Take an *odd* number for the *less side* about the rt ∠ Subtract *unity* from the square of it, and half the remainder, this will give the *greater* side about the rt ∠ For the *hypotenuse* add *unity* to the *greater* side

---

Rule of Plato

Take an *even* number for *one* of the sides about the rt ∠ From the square of *half* of this number, subtract *unity* for the *other* side about the rt ∠, and to the square of *half* this number, add unity for the *hypotenuse*

PROPOSITION XLVIII

I. 48 is the converse of I. 47, and is proved directly.

I. 48 may be extended thus —

The vertical ∠ of a △, is < than, = to, or > than a rt ∠, according as the squares on the base is < than, = to, or > than the *sum of the squares on the sides.*

From *B*, draw *BD* ⊥ to *AB*, and = *BC*, join *AD*. The square on *AD* = the squares on *AB* and (*BD* or *BC*). The st line *AC* is <, =, or > *AD*, according as the square on the line *AC* is <, =, or > the squares on the sides *AB* and *BC*. But ∠*B* is <, =, or > a rt. ∠, according as the side *AC* is <, =, or > *AD* (I. 25 and I. 8)

List of Converse Propositions

Proposition	I. 6	is the *converse* of the 1st part of	(I. 5)
"	I. 8	..	I. 4
"	I. 14	..	I. 13
"	I. 17	..	12th Axiom
"	I. 25	..	I. 24
"	I. 29	..	I. 27 and I. 28
"	I. 39	..	I. 37
"	I. 40	..	I. 38
"	I. 48	..	I. 47

Divisions of Book I

Book I, may be divided into **three** parts

The *first* part (from 1 to 26) treats of the *origin and properties of* △*s with respect to their sides and* ∠*s*, and the *comparison* of these mutually, both with regard to *equality* and *inequality*

The *second* part (from 27 to 34), treats of the *properties of parallel lines* and of ▭s

The *third* part (from 35 to 48), exhibits the *connection of the properties of* △*s and* ▭*s*, and the equality of the squares on the base and ⊥ of a rt ∠d △, to the square on the hypotenuse.

Questions on Book I.

1 Define —Postulates ; Axioms ; Problems , Theorems , Direct and Indirect methods of demonstration , Corollary , *Congruent* figures , Converse Propositions , Conterminous sides , Complementary and Supplementary angles , Internal and External segments of a line , *Proof by Exhaustion* , Exterior and Interior ∠s , Alternate and Vertical ∠s , *Altitude* of a △ ; Altitude of a ▭, *Medians* , Complements of a ▭ about the diagonal,

Analysis and Synthesis; *Orthogonal Projection*, Superposition, Magnitude, Perimeter, Dimension, Convex and Regular figures, Reflex angles; *Arithmetic mean between two-lines, Concurrent* lines, *Collinear* points, *Centroid, Ortho-centre*; *Pedal* Δ, and *Locus*

2-20 Give the *Alternative Proofs* of the following Propositions —

I 8, I 9, I 10, I 13, I 15, I 17, I 18; I 20, I 23; I 24; I 26, I 28, I 30, I 31, I 32, I 36, I 41, I 46 and I 47

21-26 Prove the *Converse* of the following Propositions —

I 5 Part 2nd, I 15, I 17, I 34, I 35 and I 41

27-31 Give the *Direct Proofs* of —

I 6, I 8, I 19, I 25, and I 40

32-39 Given the *Different Cases* of —

I 2, I 5, I 9, I 16, I 24, I 32, I 41 and I 47

40-41 Of how many parts a Δ is composed? How many kinds of Δs are there, according to the variation both of the ∠s and of the sides?

42 When are the ∠s said to be of *same species*?

43 If lines being produced ever so far both ways do not meet, can they be otherwise than |||?

44-45 Of what two parts does the *enunciation* of a *problem* and of a *theorem* consist? Into how many parts may a proposition, when complete, be divided?

46-47. Produce the lesser of the two lines = to the greater Shew that I 11 is a particular case of I 9

48-49 When are two Δs said to be equal in every respect? *What are the conditions of the equality of two Δs?*

50-53 Give a list of the *Converse* Propositions in Book I How are converse propositions generally proved? Are converse propositions universally true? Point out some converse proposition, that is not proved indirectly

54 Into how many equal parts, can an ∠ and a *straight* line be divided?

55-58 Combine (I 18 and I 5), and (I 6 and I 19) into one enunciation Combine (I 4, I 5, I 24, I. 25) into one, and (I 39 and I 40) into one

59 Mention the Propositions of Book I in which Euclid proves *the equality of two Δs, in every respect*

60-61 Mention the Props relating to the *parallelism of st lines* Enunciate the Props in which the *equality of two areas* which cannot be *superposed* on each other, is considered

62 Give the *rule* for finding the *value* of an ∠ of a *regular figure*

63 Prove Simson's 1st cor to I 32, by joining one vertex of the rectilineal figure, to each of the other vertices

64 Prove Simson's 2nd cor to I. 32, by drawing through any angular point, lines ||l to all the sides

65-67 Generalise I 41, I 47, I 48

68 Convert a *rectilineal figure* into an equivalent Δ

69 Adduce instances of *loci* from Book I

70-74 What is the reason for stating in the enunciation of I. 22, that the sum of every two of the given lines must be > than the third? Prove that when that condition is fulfilled, the two ⊙s *must intersect* Under what conditions would the ○s *not intersect*? If the sum of the lines were = to the third, would the ○s meet? Prove that they *would not intersect*

75-76 What is the *subject matter of Book I*? Into how many sections, may the Book I be possibly divided?

77-78 In order to construct a *line*, how many *conditions* must be given What problems on the *drawing of lines*, occur in Book I?

79 How many *conditions* are required in order to describe a *circle*?

80 How many conditions are necessary *to fix the position of a point in a plane*?

81. Classify the properties of Δs and ▭ms, proved in Book I.

82 Mention some propositions in Book I which are *particular cases* of more *general ones* that follow

83. State the *rule* for finding out the side of a rt. ∠d Δ, when two other sides are given

84. How many *conditions* must be given, in order *to construct a triangle*?

85 Is it possible to construct a *rectilineal figure*, the sum of whose ∠s, is an *odd number of rt ∠s*?

86-92. Show the necessity of the following *restrictions*, in the enunciations of the following propositions —

"On the side *DE* remote from *A*, describe an equilateral Δ" in I. 9 "Unlimited length" in I 12, "The opposite sides" in

I 14, "If from the ends of the side" in p 21 "*DE* is that side which is not > than the other" in I 24, "*AB* and *CD* shall be in the *same plane*" in I 27, "Towards the same parts" in I 33

93 96 What *axiom* is assumed by Euclid in I. 1, in I 20; in I 25, in I 48

97-98 State the rule of (1) *Pythagoras*, and that of (2) *Plato*, in connection with I 47.

99 Prove the following —

(1) Equal □ms on the same base and on the same side of it, are between the same ||ls

(2) If two Δs have two sides of the one respectively = two sides of the other, and the contained ∠s *supplemental*, the two Δs are = in *area*

(3) If a □m and a Δ be on *equal* bases, and between the same ||ls, the □m is double of the Δ

(4) Describe a Δ that shall be = to a given □m, and have one of its ∠s = to a given ∠

100 In Euc. I 47, why is it necessary to prove that, one side of each square described upon each of the sides containing the rt ∠, should be in the same st line with the other side of the Δ?

End of Book I

BOOK II

THEORY OF RECTANGLES

PROPOSITION I

If the line *AB* be considered as the *sum* of the *several* lines *AC*, *CD*, *DB* etc II 1 may be otherwise enunciated —The rectangles under one line and several others, is = the rectangle under that line and the sum of the others

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PROPOSITION II

In II 2, there are *three* lines to be considered (1) the *whole* line, (2) its *greater* part, (3) its *lesser* part, and here, the square on the 1st is compared with the rectangles under it and the 2nd and 3rd

If the two parts be considered as *two independent lines*, the *whole* line must be considered as their *sum*. Under this view, II. 2 may be thus enunciated—The square on the sum of any two lines=the rectangles under the sum and each of them

If the whole line AB be considered as the *greater* of two given straight lines, and one of the parts AC as the *less*, the other part BC must be their *difference* Under this view, II 2 may be thus enunciated.—The square on the greater of two lines=the rectangle under those lines together with the rectangle under the *greater* and *difference*

II. 2 is a *particular case* of II 1; for here, the two lines (see fig of 1st) P and AB are equal, and one of them AB is divided into two parts, thus:—$P \cdot AB = P \cdot AC + P \cdot BC$

Substituting AB for P, we have $AB \cdot AB$ or $AB^2 = AB \cdot AC + AB \cdot BC$, which is II 2

Alternative Proof

$AB = AC + CB$ (1), and $AB = AB$ (2) Multiplying, we get $AB^2 = AB \cdot AC + AB \cdot BC$

N B Here, the *divided* and *undivided* lines are equal

PROPOSITION III.

II 3 may be otherwise enunciated thus —

(1) The rectangle under the sum of two lines and one of them =the square of *that one* together with the rectangle under the lines.

(2) The rectangle under the two lines=the square of the *less* together with the rectangle under the *less* and the *difference*

Here, the whole line=the *greater*; one part=the *less* and the other part=the *difference*

(3) If a st line be divided *externally*, the square on the st line=the difference of the rectangles contained by the st line and the two parts

II. 2 and II. 3 may be combined into one enunciation thus:—

The difference between the rectangle under two lines and the square on one of them, is the rectangle under that one and their difference

If that one be the *greater*, this is II 2; and if it be *less*, it is II 3

II 3 is a *particular case of* II 1, for here, one of the two parts of the divided line = the undivided line

In the construction of the Text Book, the part AC is taken, the other part BC of the line AB may also be taken, and it is equally true that,

$AB \cdot BC = BC^2 + AC \cdot CB$

Alternative Proof

$AB = AC + CB$ (1), and $AC = AC$ (2), multiplying, we get $AB \cdot AC = AC^2 + AC \cdot CB$

PROPOSITION IV

II 4, may be extended thus —The square on a st line, will be = the sum of the squares on all the parts together with twice the rect under every distinct pair of them

If a straight line AB be divided into any number of segments AC, CD, DB, the squares on the whole line is = in area of the sum of the squares on the segments, together with twice the rectangle under each pair of segments

On AB construct the square $AEFB$, and join EB, through C and D, draw CG and DH ||| to AE, and through P and O draw KM and IL ||| to AB The squares KG, NQ, and DL are respectively constructed on AC, CD, and DB, and that the rectangles IP and PH are $= 2\,AC \cdot CD$, that the rectangles CO and OM are $= 2\,CD \cdot DB$, and that the rectangles AN and QF are $= 2\,AC \cdot DB$, $AB^2 = AC^2 + CD^2 + DB^2 + 2AC \cdot CD + 2CD \cdot DB + 2\,AC \cdot DB$

II 4 may be otherwise enunciated thus —The square on the sum of any two st lines = the sum of their squares together with twice the rectangles under them

Alternative Proofs

1 $AB^2 = AB \cdot AC + AB \cdot BC$ (II 2) $= (AC \cdot CB + AC^2) + (AC \cdot CB + BC^2)$ (II 3)

$= AC^2 + BC^2 + 2AC \cdot CB$

2 On AC describe the sq $ADEB$ (I 46) From AD, DE, and EB, cut off parts AX, DY, EZ each $= BC$ (I 3) Join CX,

XY, YZ and ZC. Now $\Delta ACX = \Delta XDY = \Delta YEZ = \Delta ZBC$ (I. 4) Through X and C draw XM and CM ||| respectively to AB and AD (I 31) It is obvious that, AM is a rectangle, and it is contained by AC and AX, or AC and CB Now $\Delta ACX = \frac{1}{2} AM = \frac{1}{2} AC \cdot CB$. $\Delta ACX + \Delta XDY + \Delta YEZ + \Delta ZBC = 4 \times \frac{1}{2} BC \cdot CB = 2 AC \cdot CB$.

Since, the above 4 Δs are equal, ∴ their *bases* are equal, ∴ the fig $CXYZ$ is *equilateral*; and $\angle XCZ$ = a rt. $\angle$, for ($\angle ACX + \angle XCZ + \angle BCZ$) = 2 rt $\angle$s (I 13) Also ($\angle AXC + \angle ACX$) = one rt. $\angle$. (cor 2, I. 32) and from the *identical* equality of Δs ACX and ZBC, we know $\angle AXC = \angle BCZ$; ∴ $\angle BCZ + \angle ACX$ = one rt. $\angle$; hence $\angle XCZ$ = one rt $\angle$, ∴ figure $CXYZ$ is a *square*, and it is $= CX^2 = AC^2 + AX^2$ (I 47) $= AC^2 + BC^2$.

Hence, the whole fig. $ADEB = AC^2 + BC^2 + 2 AC \cdot CB$

From II 4, a proof of I 47, may be deduced thus —

In the fig II 4, from DE and EB cut off parts DL and EM each $= BC$, join CH, HL, LM and MC Then applying a similar proof as *above*, it is obvious that, $\Delta CAH = \Delta HDL = \Delta LEM = \Delta MBC$ (I. 4), and these *four* Δs are together $= 2AG = AG + GE$, since, complement AG = complement GE Also $CHLM$ can be proved, as in the above, to be a *square* on CH

Thus $(AG + GE) + (CK + HF) = ADEB = (\Delta CAH + \Delta HDL + \Delta LEM + \Delta MBC) + HLMC$

But $AG + GE$ = these four Δs; ∴ $CK + HF = HLMC$
i e $BC^2 + AC^2 = CH^2$; or $AH^2 + AC^2 = CH^2$.

II 4. Cor —The square on a line = 4 times the square on its half, and = 9 times the square on its third part.

PROPOSITION V.

In II 5, the given finite line is supposed to be divided into two points *equally* and *unequally* Hence, several distinct linear magnitudes are to be considered, *viz.*, the *whole* line, the *equal* segments, the *unequal* segments, the *intermediate part* or *the* part *intercepted between the points of equal and unequal section*

Between these several lines, there are some obvious and important relations The whole line AB is the sum of the unequal segments AQ, QB, and each of the segments AP or PB is *half* the sum of the unequal segments Again, since the *greater* segment AQ exceeds *half* the line by the intermediate part PQ, and the half line exceeds the *lesser* segment BQ, by the intermediate

part, it follows that the *greater* segment exceeds the *lesser* segment by *twice* the intermediate. Hence it appears that, the intermediate part PQ is half the difference of the unequal parts AQ, QB.

Alternative Proofs.

(1) The line PB is divided into any two parts in Q, $\therefore$ $PB \cdot BQ = BQ^2 + PQ \cdot QB$ (II. 3).

But $PB \cdot BQ = AP \cdot BQ$ (for $AP = PB$), $\therefore$ $AP \cdot BQ = BQ^2 + PQ \cdot QB$. To each of these equals, add $PQ \cdot BQ$.

$AP \cdot BQ + PQ \cdot QB = BQ^2 + 2PQ \cdot QB$ (1). Again AQ is divided into any two parts at P, and BQ is an undivided line,

$\therefore$ $AQ \cdot QB = AP \cdot BQ + PQ \cdot QB$ (II. 1) (2), from (1) and (2), we have $2PQ \cdot QB + BQ^2 = AQ \cdot QB$.

To each of these equals, add PQ^2.

Then $2PQ \cdot QB + PQ^2 + PQ^2 = AQ \cdot QB + PQ^2$.

But $2PQ \cdot QB + BQ^2 + PQ^2 = PB^2$ (II. 4).

$PB^2 = AQ \cdot QB + PQ^2$.

(2) $\left.\begin{aligned} AQ &= AP + PQ = PB + PQ \\ BQ &= \qquad\qquad\;\; PB - PQ \end{aligned}\right\}$

$\therefore$ $AQ \cdot QB = (PB + PQ)(PB - PQ) = PB^2 - PQ^2$.

To each of these equals add PQ^2, $AQ \cdot QB + PQ^2 = PB^2$.

The Corollary to II. 5, given in p. 129 Text, is very important.

II. 5 may be otherwise enunciated thus —

(1) The rectangle under (*i. e.* contained by) any two lines together with the square on half their difference, is = the sqr. on half their sum. Here AQ and QB are considered as two independent lines, and PQ is *half their difference*.

(2) The rectangle under the sum and difference of two lines together with square on the *less*, is = to the square on the *greater*. AP and PQ are considered as two independent lines, $AQ = AP + PQ$, and $BQ = PB - PQ = AP - PQ$.

N. B. When a st. line is divided into two parts, the rect. contained by the two parts, in the *greatest* possible, and the sum of the squares of the two parts is the *least* possible, when the two parts are equal, or *when the line is bisected*.

PROPOSITION VI.

II. 6 differs only in appearance from II. 5.

In II. 6, the line AB is cut *externally* at Q, and AQ, QB are the two external segments, PB is their difference (for $AQ - QB =$

$AB=2PB$), and the intermediate part PQ is half their sum, since $AQ=AP+PB$

$\therefore AQ+QB=AP+BQ+PQ=PB+BQ+PQ=2PQ.$

See *Alternative enunciation* No (1), of II 5 above.

Alternative Proofs

(1) $\left.\begin{array}{l} AQ=AP+PQ=PQ+PB \\ BQ \qquad\quad =PQ-PB \end{array}\right\}$

$\therefore AQ\ QB=(PQ+PB)(PQ-PB)=PQ^2-PQ^2.$

To each of these equals, add PB^2, $\therefore AQ\ QB+PB^2=PQ^2$

(2) Let the st line AB be bisected at P, and produced to Q Produce QA to R, making $PR=PQ$ Now $RP+PB=QP+AP$ *i.e* $RB=AQ$ (Ax. 2) Now the line RQ is divided into two equal parts in P, and into two unequal parts in B, $\therefore$ $RB.\ BQ+PB^2=PQ^2$ (II 5) But $RB=AQ$ (proved), $\therefore AQ.\ BQ+PB^2=HQ^2$.

II. 5 and II 6, may be included in one enunciation thus .—

If a st. line be bisected, and also divided (*internally* as in II. 5) or (*externally* as in II. 6) into two unequal segments, the rectangle contained by the unequal segments is = to the difference of the squares on *half* the line, and on the line between the points of section

Proof—Let AB be the bisected at P, divided unequally *internally* or *externally* at Q Now AQ is the *sum* of AP and PQ, and BQ is their *difference*, since $AP=BP$, $\therefore AQ.QB$ is the rectangle contained by the sum and difference of AP, PQ, and it is $\therefore$ = to the difference of the squares on AP and PQ (II 5, cor).

PROPOSITION VII.

Either of the two parts AC, CB of the line AB, may be taken; it is equally true that, $AB^2+AC^2=2\ AB.\ AC+BC^2$.

II 7 may be otherwise enunciated thus .—

(1) The sum of the squares on any two st lines is = to twice the rect. under them, together with the square on their difference.

(2) The square on the difference of any two lines = the difference between the sum of squares on the two lines, and twice their rectangle

N B. In the above two cases, AB, BC are considered as two independent lines, and AC = their difference

Symbolical form of proof of II 7.

$AK = CE = AB \cdot BC$, $AK + CE = 2AB \cdot BC$.

$(AG+CK)+(CK+GE)$ or $(AG+CK+GE) + CK = 2AB.BC$

$$\left.\begin{array}{l} i\ e\ \text{Gnomon } AKF + BC^2 = 2\,AB \cdot BC \\ \qquad HF \quad = AC^2 \end{array}\right\}\text{Adding, we have}$$

$$(AKF + HF) + BC^2 = 2AB \cdot BC + AC^2$$

$$\therefore AB^2 + BC^2 = 2\,AB \cdot BC + AC^2$$

Alternative Proofs of II 7

(1) $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$ (II 4) Add to both CB^2, $\therefore AB^2 + BC^2 = AC^2 + (2 \cdot AC\,CB + 2BC^2)$, but $AC\,CB + BC^2 = AB \cdot BC$ (II 3), and doubles of these are equal, $\therefore 2AC.\,CB + 2BC^2 = 2AB \cdot BC$, $AB^2 + BC^2 = 2AB \cdot BC + AC^2$

(2) Describe a sq $APQC$ on AC (I 46) Produce PA to L, making $AL = CB$, produce QP to M, making $PM = CB$ produce CQ to N, making $QN = CB$ Join LB, BN, NM and ML The figure $LMNB$ is *equilateral*, its *sides* are the *bases* of *four* equal Δs LAB, MPL, NQM and BCN (I 4), and it is a *rectangular* ▭m, in $\Delta\, LPM$, ($\angle\, PLM + \angle\, PML$) = 1 rt $\angle$, and in $\Delta\, ALB$, ($\angle ALB + \angle ABL$) = 1rt $\angle$, and since ΔLPM is identically = ΔALB $\angle ALB = \angle PML$, and from above, we have ($\angle PLM + \angle PML$) = ($\angle ALB + \angle ABL$), of which $\angle ALB$ being = $\angle PML$, the remaining $\angle\, PLM = \angle ABL$, hence, $\angle PLM + \angle ALB$ = 1 rt $\angle = \angle MLB$ Similarly, $\angle LMN$ can be proved = a rt $\angle$, . $\angle MLB + \angle LMN$ = 2 rt $\angle$s, LB is ‖l to MN, so LM can be proved ‖l to BN Thus the figure $LMNB$ being equilateral rectangular ▭m, is a *square* Through B, and N, draw BX and NX ‖l respectively to CN and CB, meeting at X, (I 31) It is obvious that $CNXB$ is a rect and = $AB \cdot BC$

$\Delta LAB + \Delta\, MPL + \Delta\, NQM + \Delta\, BCN = 2\, CNXB = 2AB\,BC$, for each $\Delta = \frac{1}{2}\, CNXB$, figure $LMNB = BN^2 = BC^2 + CN^2$ (I 47) $= BC^2 + AB^2$. Again, $LMNB = 4\, \Delta$s $+ AC^2 = 2AB \cdot BC + AC^2$; $AB^2 + BC^2 = 2\,AB \cdot BC + AC^2$

PROPOSITION VIII

II. 8 may be otherwise enunciated thus —(1) The square on the sum of two lines = four times the rectangle under them, together with the square on their difference

(2) The square on the sum of two lines AB, BC, exceeds the square on their difference AC, by *four* times the rectangle contained by them

N B. Either part of line AB, may be taken, and it is also true that $4AB.AC+BC^2$=sq on the line made of AB, AC together

Euclid's proof of II 8.

Produce AB until BD is $=CD$ (I 3 and Post 2) On AD construct the square $AEFD$ (I 46) and join ED, through C and B, draw CH and BL ||| to AE (I 31), and through K and P, draw MN and XO ||| to AD (I 31).

$\because$ GK is $= CB$ (I 34), $CB= BD$ (Constr), and $BD=KN$ (I 34), $GK = KN$ (Ax 1) and $\therefore$ the rectangle GL=rectangle KF, and $\therefore$ AK and KF are complements, they are equal (I 43), $\therefore$ $AK = GL$ (Ax 1), $\because$ $GK=KN$, $\therefore$ $GR = BN$ (I 46, cor. 1), and MP and PL are complements, they are equal (I 34); adding these equals together, the rectangle $MP+NB^2$=the rectangle GL (Ax 2), and $\therefore$ =rectangle AK (Ax 1), $\therefore$ $AK+GL+KF+MP+BN$=*four* times AK, but $AK+GL+KF+MP+BN+XH^2$, make up the whole square $AEFD$, $\therefore$ the square $AEFD$ is $=4(AK+XH)$ (Ax 1) But $\because$ $BK=BD$ (II 4, cor 4) and $BD=CD$ (Constr), $BK=CB$ (Ax 1), and $\therefore$ AK is the rectangle under AB and CB, and $XP=AC$ (I 34), $\therefore$ XH is the square on AC, the square on the sum of AB and BD is $=4\,AB.\,CB+AC^2$

Alternative Proofs.

(1) $\because$ AD is divided into any two parts in B, $AD^2= AB^2+BD^2+2AB\,BD$ (II 4) But $CB=BD$, $\therefore$ $AD^2=AB^2+BC^2+2\,AB\,BC$ Again, since AB is divided into any two parts, $\therefore$ $AB^2+BC^2=2\,AB\;BC+AC^2$ (II 7).

$\therefore$ $AD^2=4\,AB.\,BC+AC^2$

(2) Since AB is divided into any two parts in C, $\therefore$ $AB\,BC= AC\,CB+BC^2$ (II 3), $\therefore$ $4\,AB\;BC=4AC\,CB+4BC^2=2AC\,CD+CD^2$, for $CD=2\,BC$; and $CD^2=4$ times the square on *half* the line CD or BC

$\therefore$ $4\;AB\,BC+AC^2=2\;AC\,CD+CD^2+AC^2$, adding AC^2 to both, $=AD^2$ (II 4).

(3) Take the **last figure** of II. 8 as given on p 133, **Text Book**

Let AB be divided at C Produce AB to D, making $BD=BC$ On AD describe a square $AEFD$ (I 46). From DF, FE and EA cut off DX, FY and EZ each$=AB$ (I 3) Through B and Y, draw BQ and YM ||| to AE or DF (I 31), and through Z and X, draw ZN and XP ||| to AD or EF (I. 31). All the □ms in the

figure, *viz*, AN, ZY, YX, QD and MQ are rectangles, ∵ DX, FY and EZ are each $= AB$, and $AD = DF = FE = EA$ (being sides of a square), ∴ $XF = YE = ZA = BD = BC$ ∴ the *four* rectangles AN, ZY, PF and XB are each $= AB\ BN = AB\ BD = AB\ BC$ Again, ∵ BQ, XP, YM and ZN are each $= AB$ (I 34), and BN, XQ, YP, and ZM are each $= BD$ or BC (I 34), ∴ NQ, QP, PM and MN are each $= AB - BC = AC$, ∴ figure MQ is *equilateral*, but it has been proved that, it is a *rectangle*, ∴ MQ is a *square*, and since, each of its sides $= AC$, ∴ $MQ = AC^2$ Now the figure $AEFD = AD^2$ and $AEFD = (AN + ZY + PF + XB) + MQ = 4\ AB\ BC + AC^2$

$$AD^2 = 4\ AB\ BC + AC^2$$

PROPOSITIONS IX and X

The *following* points should be especially remembered in proving II 9 and II 10 —

(1) The four Δs APC, BPC, CED, and DQB are *isosceles* (2) $\angle ACB$ is a rt $\angle$ (3) In an isosceles rt $\angle$ d Δ, each of the acute $\angle$s $= \frac{1}{2}$ a rt $\angle$

Symbolical form of the proof of II 9 or II 10

$AQ^2 + BQ^2 = AQ^2 + QD^2 = AD^2$ (I 47) $= AC^2 + CD^2$ (I 47) $= (AP^2 + PC^2) + (CE^2 + DE^2) = 2\ AP^2 + 2\ ED^2 = 2\ AP^2 + 2\ PQ^2$

II 9 may be otherwise enunciated —(1) The sum of the squares on any two st lines = twice the square on half their sum, together with twice the square on half their difference (2) The square on the *sum* and the square on the *difference* of two st lines = twice the sum of the squares on the two st lines

Alternative proof of II 9

AB is divided into any two *unequal* parts at Q, and *equally* P Also AQ is divided into any two parts at P, ∴ $AQ^2 = AP^2 + PQ^2 + 2\ AP.PQ$ (II 4) Add BQ^2 to each, ∴ $AQ^2 + BQ^2 = AP^2 + PQ^2 + 2\ AP\ PQ + BQ^2 = (BP^2 + PQ^2) + (2\ BP\ PQ + BQ^2)$, (for $AP = BP$), now since BP is divided into any two parts at Q, ∴ $BP^2 + PQ^2 = 2\ BP\ PQ + BQ^2$ (II 7)

$$\therefore AQ^2 + BQ^2 = 2\ BP^2 + 2\ PQ^2 = 2\ AP^2 + 2\ PQ^2$$

Alternative proofs of II. 10

(1) Let AB be divided into two equal parts at P and produced to Q Produce PA to R making $PR = PQ$ Thus we have $RP + PB = QP + PA$ or $RB = QA$, ∴ RQ is divided equally at P

and unequally at B, $\because RB^2+BQ^2=2RP^2+2PB^2$ (II 9) But $AB=QA$, $RP=QP$ and $PA=PB$; $QA^2+BQ^2=2QP^2+2AP^2$

(2) Let AB be bisected at A and produced to Q, QP^2+BQ^2 shall be $=2AP^2+2BQ^2$, $\because$ since AQ is divided into any two parts in P, $\because AQ^2=AP^2+2AP \cdot PQ+PQ^2$ (II 4), add BQ to each

$\therefore AQ^2+BQ^2=AP^2+PQ^2+2AP \cdot PQ+BQ^2=PB^2+PQ^2+2PB \cdot PQ+BQ^2$ for $(AP=PB)$ But since PQ is divided into any two parts in B, $PQ^2+PB^2=2PB \cdot PQ+BQ^2$ (II. 7), $AB^2+BQ^2=(PB^2+PQ^2)+(PQ^2+PB^2)=2PB^2+2PQ^2=2AP^2+2PQ^2$

II 9 and II 10, may be included under one enunciation thus —

(1) If a st line be divided into two *equal* parts, and also into two *unequal* parts, either *internally* (as in II. 9) or *externally* (as in II 10), the squares on the *unequal* parts are together *double* of the squares on *half* the line bisected, and on the line between the points of section (See Cal Ex Pap. 1875 q 3)

Or (2) The sum of the squares on two st lines = twice the sum of the squares on *half* their *sum* and on *half* their *difference*

Or (3) The square on the sum of two lines and the square on their difference, are together = *double* the sum of the squares on the two lines

Draw *two* figures In the 1st., let AB be divided equally at P and unequally at Q In the 2nd, let AB be divided equally at P, and produced to Q Thus AB is divided into two unequal segments *internally* at Q (in fig 1st) and *externally* at Q (in fig 2nd) From both of these figures, we have, (1) $AQ^2=AP^2+PQ^2+2AP \cdot PQ$ (II 4), and $PQ^2+PB^2=2PQ \cdot PB+BQ^2$ (II 7) or $BQ^2+PB \cdot PQ=PB^2+PQ^2$ (2) i e $BQ^2+2AP \cdot PQ=AP^2+PQ^2$ (for $PB=AP$) Adding (1) and (2), we have $(AQ^2+BQ^2)+2AP \cdot PQ=(2AP^2+2PQ^2)+2AP \cdot PQ$ Taking $2AP \cdot PQ$ from both sides, we get $AQ^2+BQ^2=2AP^2+2PQ^2$

N B The *first ten propositions* of B II, contain the *whole theory of the relations of the rectangles and squares on divided lines and their parts*

PROPOSITION XI

Symbolical form of the proof of II 11.

$CF \cdot FA + AE^2 = EF^2$ (II 6) $= EB^2 = (EA^2 + AB^2)$

Taking away the common part AE^2 from both, we have $CF \cdot FA = AB^2$ i e figure $FCKG = AB^2$ or $(CH + AG) = (CH + DH)$

Taking away CH, we get $AG = DH$ or $AH^2 = BD \cdot BH = AB \cdot BH$, $\therefore AB \cdot BH = AH^2$

For, Medial section of a line—see p 139 Text.

N B A line divided as in II 11, is said to be divided in "extreme and mean ratio"

II 11 is a case of the following problem —In a given straight line or its continuation, to find a point such that the rectangle under the whole line and the segment between one of its extremities and that point, shall be = in *area* to the square on the segment between its other extremity and that point

II 11 may be otherwise enunciated thus —

(1) Cut a line in "*extreme and mean ratio*," *internally* or *externally*

(2) Divide a line *internally* or *externally* into *medial* section The 1st case of the above problem, is solved by Euclid in II 11 The solution of the 2nd case, *viz*, The external division of a st line in medial section is given below (See Text Ex 21 p 146)

Describe a square $ABCD$ on the given line AB (I 46) Bisect BC at E (I 10), join AE Produce BC to F, making $EF = AE$ On BF, describe a square $BHGF$, which will have a side BH in the same direction as BA. Produce DC to meet HG in K. BC is bisected at E, and produced to F, $BF \cdot FC + BE^2 = EF^2$ (II 6) $= AB^2 + BE^2$ Taking away BE^2 from both we have $BF \cdot FC = AB^2$ Now $BF \cdot FC$ = figure FK (for $FG = BF$) Add figure CH to each, thus, we have $(FK + CH) = (AB^2 + CH)$ i e. $FH = DH$ or $BH^2 = AB \cdot AH$ i e $AB \cdot AH = BH^2$

N B—In order to cut a line in '*extreme and mean ratio*,' it is 1st, necessary *to produce it in extreme and mean ratio*, i e *to produce it so that, the rectangle under the whole line thus produced*, and the produced *part shall be = square on the line itself*

From the proof of II 11, it appears that $CF \cdot FA = CA^2$

CA has been produced to F in this way, and $CA = AB$

Considering CF as a line cut in "*extreme and mean ratio*" at A, it will appear, that rectangle under the *greater* segment, and

difference of the segments=square on the *lesser* segment, for AC=*greater* segment, and $=AB$; AF (which is $=AH$) is the *less*, HB=difference of the segments From II '11, we know $AB.BH=AH^2$

Hence it appears, that :—

If a line be cut in "*extreme and mean ratio*," the *greater segment* will be cut in the same manner, by taking on it, a part= the less And the less will be similarly divided, by taking on it a part=the difference, and so on

In II 11, it is taken for granted that, if $CF.FA=CA^2$, then $CA>AF$ It may be proved thus —$CF.FA=CA.FA+FA^2$, $\therefore$ CA^2 exceeds AF^2 by $CA.AF$, and $\therefore$ CA must be $>AF$

Cor. If a line be cut in "*the extreme and mean ratio*," the rectangle under the segments=difference between their squares

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PROPOSITION XII.

Symbolical form of the proof of II 12

$BD^2=BC^2+CD^2+2BC.CD$ (II 4). By adding AD^2 to each, $AD^2+BD^2=BC^2+(CD^2+AD^2)+2BC.CD$

$\therefore$ $AB^2=BC^2+AC^2+2BC.CD$

It is evident that, if the $\perp$ were drawn from B to AC produced, it would in like manner be proved, that twice the rectangle under AC and its production, would be=to the excess of the square on AB, above the squares on AB and BC. And hence it follows, that the rectangle $BC.CD$ is =the rectangle under AC, and its produced part

Converse of II. 12.

If the square described on one side of a Δ, be $>$ than the sum of the squares described on the other two sides, the $\angle$ opposite to the first side, is *obtuse*

In ΔABC, if $AB^2>(AC^2+CB^2)$, then shall $\angle ACB$ be *obtuse* If $\angle ACB$ be not *obtuse*, it must be either *right* or *acute* If $\angle ACB$ be *a rt* $\angle$, then $AB^2=AC^2+CB^2$ (I 47), but it is not; $\therefore$ $\angle ACB$ is not a rt. $\angle$ If $\angle ACB$ be *acute*, then AB^2 would be *less* than (AC^2+CB^2), (II 13), but it is not, $\therefore$ $\angle ACB$ is not an *acute* $\angle$, $\therefore$ $\angle ACB$ is *obtuse*

N. B If the *obtuse* $\angle$ become more and more *obtuse*, till at length the vertex of the Δ falls on the base produced, then II 13 becomes II 4.

PROPOSITION XIII

Symbolical form of the proof of II. 13

Let ABC be a Δ, having $\angle B$ an *acute* $\angle$ and let AD be a $\perp$ from A to the opposite side BC

Then AC^2 shall be $= AB^2 + BC^2 - 2\ CB\ BD$.

Because, BC is divided *internally* and *externally*, into any two parts BD, DC

$\therefore\ CB^2 + BD^2 = 2\ CB\ BD + CD^2$ (II 7)

By adding AD^2 to both, we get $CB^2 + (BD^2 + AD^2) = 2CB\ BD + (CD^2 + AD^2)$, *i e*, $CB^2 + AB^2 = 2CB\ BD + AC^2$ *i e*, $CB^2 + AB^2 - 2CB\ BD = AC^2$

Again, if the Δ be *right* $\angle$d, then $AB^2 = AC^2 + BC^2$ (I. 47), add BC^2 to both sides, then $AB^2 + BC^2 = AC^2 + 2BC^2$, $\therefore AB^2 + BC^2 - 2\ BC\ BC = AC^2$

N B From B and C, drop two $\perp$s BY and CY, on AC and AB, respectively

(1) If $\angle A$ be *acute*, then $BC^2 = AB^2 + AC^2 - 2\ AB\ AX$ (II 13), and $BC^2 = AB^2 + AC^2 - 2AC\ AY$ (II 13) Hence, we have $AB\ AX = AC\ AY$

(2) If $\angle B$ be *acute*, $AC^2 = AB^2 + BC^2 - 2\ CB\ BD$ (II 13) and $AC^2 = AB^2 + BC^2 - 2AB\ BX$ (II 13)
Hence, $CB\ BD = AB\ BX$

(3) If $\angle C$ be *acute*, $AB^2 = AC^2 + CB^2 - 2CB\ CD$ (II 13) and $AB^2 = AC^2 + CB^2 - 2\ AC\ CY$ (II 13)
Hence $BC\ CD = AC\ CY$.

II 12 and II 13 may be included under one enunciation thus —

The difference between the square on one side of a Δ, and the sum of the squares on the other two sides, is $=$ twice the rectangle under either of these two sides, and the *intercept* between the $\perp$ on it, and the $\angle$ included by the sides

Converse of II 13

If the square described on one side of a Δ, be $<$ than the sum of the squares described on the other two sides, the $\angle$ opposite to the first side, is *acute*

In $\Delta\ ABC$, if AC^2 is $< (AB^2 + BC^2)$, then $\angle ABC$ shall be *acute*

If $\angle ABC$ be not *acute*, it must be either *right or obtuse* If $\angle ABC$ be a rt $\angle$, then $AC^2 = AB^2 + BC^2$ (I 47), but, it is not,

∴ ∠ ABC is *not* a rt. ∠. If ∠ ABC be *obtuse*, then AC^2 would be > (AB^2+BC^2) (II 12), but it is not; ∴ ∠ABC is not an *obtuse* ∠, ∴ ∠ ABC is *acute*

N. B. If the *acute* ∠ become more and more acute, till at length the vertex of the Δ, falls on the base or base produced, then II. 13 becomes II. 7.

I. 47. II 12, and II. 13, may be combined into one enunciation :—

The square on any side of a Δ, is > than (II. 12), = to (I 47) or > then (II. 13), the squares on the other two sides, according as the ∠ opposite to that side is > than, — to, or < than a rt. ∠. The difference is twice the rectangle contained by either of the other sides and the straight line intercepted between the vertex of that ∠ and a ⊥ draw to the remaining side from its opposite ∠.

I 47, II 12, and II 13 show the relations between the sides of a *right* ∠d Δ, *obtuse* ∠ d Δ, and an *acute* ∠ d Δ, respectively.

PROPOSITION XIV

Symbolical form of the proof of II. 14.

$BE \cdot EF + GE^2 = GF^2 = GH^2 = GE^2 + EH^2$.

∴ $BE \cdot EF = EH^2$ *i e* rectangle $BD = EH^2$ *i. e.* rectilineal figure $A = EH^2$.

From II 14, it appears that, if a ⊥ BA be drawn from any point in a semi ◡ to the diameter, the square on the ⊥ is = to the rectangle under the segments, into which it divides the diameter.

Def. The process of finding a square which is = the area of a given figure, is called the **quadrature of the figure.**

N. B.—II. 14, is the *fourth* step in the *quadrature* of a given rectilineal figure—the previous steps being—I. 47, I. 44, I. 45.

The "Subject-matter" of Second Book of Euclid, is the theory of rectangles

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QUESTIONS ON BOOK II.

101-2 What is the "Subject-matter" of Book II. Define a *gnomon*

103-4 When is a line said to be divided *internally*? When *externally*?

105-16 Give the *Alternative Enunciations* of —

1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th, 9th, 11th, 12th, 13th propositions, Book II

117-25 Give the *Alternative Proofs* of —

2nd, 3rd, 4th, 5th, 6th, 7th, 8th, 9th, 10th propositions, Book II.

126 How is a line divided, so that the rectangle contained by its segments, may be *maximum* (*greatest* possible)

127 How is a line divided so that the sum of the squares on its segments, may be *minimum* (*least* possible)

128 Divide a line *externally* in *medial* section; or cut a line *externally* in *extreme and mean ratio*

129 Deduce a proof of I 47 from II 4

130-36 Give the *Symbolical forms of Proofs* of .—

7th, 9th, 10th, 11th, 12th, 13th, 14th propositions, Book II.

137-38 Combine the following *Propositions into one* —

II 5 and II 6, II 9 and II 10

139-40 Compare I 47, II 12, and II 13 Combine the enunciations of these *three* propositions, into one

141-42 Shew that II 2 and II 3 are *special cases* of II 1

143 Give the corresponding *Algebraical formulæ*, of all the propositions of Book II

144 The *difference* of the squares on two st lines is = the rectangle contained by their *sum* and *difference* Prove this

145 Give Euclid's proof of II 8

146 Prove II 4 from II 2 and II 3

147 Prove II 6 from II 5

148 Shew that, if a line be divided into two equal parts, and into two unequal parts, the part of the line between the points of section is = half the difference of the unequal parts

149 If half the sum of two unequal lines, be *increased* by *half* their *difference*, the *sum* will be = to the *greater* line, and if the sum of two lines, be *diminished* by *half* their *difference*, the *remainder* will be = to the *less* line

150-52 State and prove the **converse** of —II 12 and II. 13 Include II 12 and II 13 under *one* enunciation

153 From II 3, shew that, the difference between the rectangles contained by the whole line *AB* and each of the parts *AC* and *BC* is = the difference of the squares on the parts, *BC*, *AC*

154-55 In how many ways, may the difference of two lines be exhibited Enunciate the propositions in Book II, which depend on that circumstance

156 Shew that, if two *complements* be together = the two squares, the given line is *bisected*

157 How may, a *series of lines be found, similarly divided to the line AB, as in II 11*

158 "*All plane rectilineal figures admit of quadrature*" Point out the succession of steps, by which Euclid establishes the truth of the above

159-60. Could any proposition of Book 2nd, be made *corollaries* to other proposition, with advantage? Point out any such propositions, and give your reasons

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BOOK THIRD

Subject matter —The subject matter of Euclid's B III, is the properties of the circle.

N B Indirect demonstration are more frequently employed in Book III, than in Book I In Book I, out of 48 propositions, *nine* are proved *indirectly*, in Book III, out of 37 propositions 15, are proved *indirectly*

PROPOSITION I

The best *practical* method of *finding the centre* of a ⊙, is to *bisect* any two chords in a ○, and at the points of bisection, to draw ⊥s to the chords, the *intersection* of these ⊥s is the *centre*

Alternative proof of III 1.

Take any two ∥ chords *AB* and *CD* in the ○ *ABC* Bisect *AB* at *E* and *CD* at *F* Join *EF* Produce *EF* to meet the ⊙ce of the ○ in *G* and *H* It is obvious that *GH* is a diameter of the ○ Bisect *GH* at *O*, and *O* is the centre of the ○ *ABC*

In the construction of III 1, *DC* is said to be *produced* to meet the ⊙ce of the *ABC* in *E* and *C*, this assumes that *D* is *within* the ⊙, which Euclid proves in III 2

If the point *G* (figure III. 1), be in the diameter *CE*, but not coinciding with its middle point, then it is evident, that *G* cannot be the centre of the ○, for *GC* is not = *GE*

Proof of the corollary to III 1.

Let the st line CE bisect a chord AB of ○ ABC at rt. ∠s at D, then CE shall pass through the centre of the ○.

CE bisects AB at rt ∠s (hyp), ∴ every point in CE is equidistant from A and B But since A and B are on the ○ce of the ○ ABC, ∴ the centre of the ○ ABC is equidistant from A and B (Def 11), ∴ CE passes through the centre

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PROPOSITION II

If AB be produced in either direction, any pt in the produced part, may be shown to be *without* the ○ Take any pt. C in AB produced Join CD ∠ABD is > than ∠ ACD (I 16), ∴ ∠BAD > ∠ACD, for ∠BAD = ∠ABD Hence $CD > DA$, but DA is the radius of the ○, ∴ the pt C is without the ○

PROPOSITION III

III 3, as given by Euclid, consists of two *distinct parts*, each the *converse* of the other III 3 is the *converse* of the Cor III 1

The *mutual relation* of the two parts of III 3, may be shown in the following manner —

Cor III 1 —If a line bisects a chord of a ○, and is ⊥ to it,	The line passes through the centre of the ○
III 3, part I —If a line passes through the centre of a ○,	The line is ⊥ to the chord
III 3, part II —If a line passes through the centre of a ○, and is ⊥ to a chord of the ○,	The line bisects the chord.

Alternative proof of III 3, part II

$EB^2 = EA^2$, (∵ $EB = EA$), and $EB^2 = BF^2 + EF^2$ (I 47), and $EA^2 = AF^2 + EF^2$ (I 47)

∴ $BF^2 + EF^2 = AF^2 + EF^2$, ∴ $BF^2 = AF^2$, ∴ $BF = AE$

N B The *restriction* that "*a chord which does not pass through the centre*" in the enunciation of III 3, is important—for a st line drawn through the centre of a ○ (a diameter for instance), may bisect another (another diameter), and yet not intersecting the second diameter at rt ∠s

N B From III 3, it appears that—

(1) If a system of ∥ chords be drawn in a ⊙, the *locus* of their points of bisection, is the diameter of the ⊙, which is ⊥ to them

(2) The st line which bisects any chord ⊥ly, bisects every chord ∥ to it ⊥ly, and is a diameter of the ○

PROPOSITION IV

It follows from III. 4, that no □m, except a *rectangle*, can be *inscribed* in a ○ For the diagonals of a □m bisect each other, and ∴ must both pass through the centre, and must ∴ be equal, each being a diameter. Hence, the □m must be a rectangle

N.B If two chords of a ○ bisect each other, they are both diameters.

PROPOSITIONS V. and VI.

III. 5 and III. 6 may be included in one enunciation thus —

If the ○ces of two ⊙s meet at *a* point (or at *two* points) they cannot have the same centre.

III. 5 may be better expressed thus :—

Concentric ⊙s cannot meet, and that which has the lesser radius, will be included within the other If the ⊙s had the same radius, they would coincide, and be the same ⊙.

The points of the ○ce of that which has the lesser radius, being less distant from the centre, than those of the ⊙ce of that which has the greater radius, must be all within the latter Consequently, the ○s cannot meet, either by contact or intersection *This proof includes* III. 6

N B It would appear as if Euclid had made *three* cases, one in which the ○s cut, one in which they touch *internally*, and one in which they touch *externally*, and had then left the *last* case evident

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PROPOSITIONS VII and VIII

III 7 and III 8 expresses the same property, in the former, the point is taken *in the diameter*, and in the latter, *in the diameter produced*, and exhibit an instance of the division of the diameter into internal and external segments

In III 7, it is assumed that, $\angle FEB >$ the $\angle \Gamma EC$, the hypothesis being only that the $\angle DFB >$ the $\angle DFC$, and that in III 8, it is assumed that, E falls within the $\Delta A\Gamma C$ and H is without ΔACG

A proposition similar to III 7 and III 8

If any point be taken *on the* ◯ce of a ◯, of all the st lines which can be drawn from it to the ◯ce, the *greatest is that* in which the centre is, and of any others, that which in nearer to the st line which passes through the centre, is always > than one more remote, and from the same point, there can be drawn to the ◯ce, *two* st lines, and only two, which are *equal* to one another, one on each side of the greatest line

N B The *first two* parts of the above, is contained in III 15 and the *third* part will be required in III 10 (See Notes on III 10)

The results of III 7 and III 8 may be expressed thus —

(1) If a line always terminated in the ◯ce of a ◯, revolve round a point F, within a ◯, different from the centre Γ, it will *vary* in its magnitude between certain *limits* As it revolves from the position FEA towards D in either direction, it diminishes, and at equal distances at each side of ΓEA it has equal magnitudes, and this diminution continues until, having made half a revolution, it assumes the position ΓD In the position ΓA, it is *maximum*, and in the position ΓD, it is *minimum*, and the nearer it is to the *maximum* position, the *greater* it is, and the nearer to the *minimum* position, the *less* it is

(2) If a line be supposed to revolve round a *fixed* point A (taken in the diameter produced), as it recedes from AD in either direction, it diminishes When it recedes so far, that the part intercepted within the ◯ vanishes, and the two points of intersection with the ◯ unite and become one, the line becomes a *tangent* If it recede beyond this, it will not meet the ◯ at all As the line revolving from the tangential position again approaches AB, being terminated in the *convex* part of the ◯ce, it still diminishes, and becomes a *minimum*, where it assumes the position AB Thus the *tangent* is *less* than any *secant* from the same point, but *greater* than the external part of the *secant*

Corollaries to III 7 and 8

(1) If a point be taken *within* or *without* a ◯, of all other st lines drawn from it to the ◯ce, the *greatest* is that which meets the ◯ce, after passing through the centre.

(2) If two chords of a ○ intersect each other, and make equal ∠s with a diameter at the point of intersection—the two chords are = to one another. This is obvious if *GF*, *HF* be produced to meet the ⌒ce in *M*, and *N*. Then *MF*=*NF*, hence *GM*=*HN* (See fig. III. 8).

(3) If two chords of a ⌒ intersect each other when produced, and make equal ∠s with the diameter produced, and passing through the pt. of intersection, the two chords may be shown to be equal (See fig. III. 8).

Let *AM* be produced to meet the ⌒ce in *P*, the chord *MP* may be shown to be = the chord *EH*.

From 2nd and 3rd corollaries above, it is evident that:—

If from any pt. within a ○ which is not the centre, st. lines be drawn to the ⌒ce—those lines which form equal ∠s with the line passing through the centre are equal; and if from any point without a ○, st. lines be drawn to the ⌒ce—those lines which form equal ∠s with the line passing through the centre, are equal.

PROPOSITION IX

III. 9, is a corollary deducible from III. 7, case (4); for it is shown in III. 7 that—only two equal st. lines can be drawn to the ⌒ce, from any point which is not the centre; but by (hyp.) from a certain point, three equal st. lines can be drawn; ∴ that pt. is not a pt. which is not the centre; *i. e.* it is a pt. which is the centre.

N. B. The *Criterion* for the "determination of the centre" is that, more than two pts. of the ⌒ce, should be equally distant from it.

PROPOSITION X

Two Additional cases of the (Second Proof) of III. 10

(1) When the centre of the ○ *DABC* is conceived to fall *without* the ⊙ *EABC*.

Let it be possible that the two ○s cut one another in *three* pts. *A*, *B*, *C*. Find *H*, the centre of the ○ *DABC* (III. 1). Join *HA*, *HB* and *HC*; then *HA*=*HB*=*HC* (Def. 11); and ∵ not more than two equal st. lines can be drawn to the ⌒ce of the ○ *EKBC* from the pt. *H*, which is *without* the ○ *EABC* (III. 8, case 5); ∴ *HA*, *HB* and *HC* are *not* equal; but it has been proved that, *HA*=*HB*=*HC*, which is absurd; ∴ one ○ can not cut another in more than two points.

(2) When the pt H is *on* the ○ce of the ○ $EABC$; then H is the centre of the ○ $DABC$, $HA=HB=HC$ (Def 11), but from H (which is on the ○ce of ○$EABC$)—*only two* equal st lines (and not more) can be drawn to the ○ce of the ○ $EABC$, (See notes on III 7,8), and since three equal st lines are drawn, it is impossible, one ○ cannot cut another at more than two pts

By III 10, two ○s cannot *intersect* in more than two pts, and that they cannot *touch* in more than two pts, hence, two ○s cannot have more than two pts *in common*

Hence it appears, that if two ○s coincide at *three* pts they will coincide at every pt, or only one ○ can be drawn through *three* given pts

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PROPOSITIONS XI AND XII

It would be better to begin III 11 and III 12 thus.—

III 11 Let ABC and AED be two ○s which touch one another *internally* at A

Let F be the centre of the larger ○ ABC Join AF, the centre of the smaller ○ is in the line AF If not, let it be in any other position such as G Join FG and GA Produce FG to meet the ○ces of the ○s in E and H (For the rest, see the Text Book)

III 12 Let ABC and ADE be two ○s which touch one another *externally* at A, let F be the centre of the ○ ABC Join FA, produce FA to meet the ○ce of the ○ ADE in E Then the centre of the ○ ADE must be in the line AE If not, let it be elsewhere, as at G Join FG intersecting the ○s in H and K Join GA (For the rest, see the Text Book)

In Euclid's enunciations of III. 11 and 12, he speaks of "*the point of contact*," thereby assuming that there is only *one* pt of contact This is proved in III 13, and ought not to have been anticipated

In order to remove this objection, III 11 and 12, may be enunciated in the following manner —

(1) If two ○s touch one another *internally*, in *any* pt the st line joining their centres, being produced, shall pass through that point

(2) If two ○s touch one another *externally*, in *any* pt the st line joining their centres, shall pass through that point.

III. 11 and 12, may be included in one enunciation, thus :—

(1) If two ○s touch each other at any pt, the centres and that pt. are *collinear.*

(2) If two ○s touch one another, their ⊙ces cannot have a common pt. out of the direction of the st line, which joins the centres

From III 11 and 12, it follows that—the line joining the centres of *contingent* ○s—is the *sum* of the radii, when the contact is *external*, and the *difference* of the radii, when it is *internal*

Let F and G be the centres of the ○s, which touch *internally* at A Join FG, which when produced, shall pass through A Then AF and AG are radii of the two ○s, and $AF=AG=FG$. Let F and G be the centres of two ○s which touch *externally* at A Join FG, which passes through A Then AF and AG are radii of the two ○s, and $AF+AG=FG$

Direct Proof of III 11 and III 12.

Let a straight line touch both ⊙s *externally* at A Let F, G be the centres of the ○s Join FA, GA, then each of the lines is a rt ∠s to the line which touches the ○s (III 18), at A, ∴ FAG is a st. line (I 14), or if one ○ touches the other *internally*, AF coincides partly with AG

N. B—Two ○s cannot cut in more than two pts (III 10)

If they cut in two pts., the line joining their centres, cannot go through either of the pts, ∵ if the line joining the centres go through a pt in which the ○s meet, the ○s must touch one another *internally* or *externally* at that point,—which is the *Converse of* III 11 and 12

Alternative Proof of III. 11.

GH is the least line, that can be drawn from G to the ⊙ce of the ○, whose centre is F (III 7, case 2), ∴ $GH<GA$ *i e*, $GH<GE$, which is absurd.

So III 12, may be deduced from III 8

PROPOSITION XIII.

In III. 11 and 12, it is proved that, the line joining the centres of *contingent* ○s—passes through a pt of contact In III. 13, it is shown that this is *the only* point of contact, by proving that an absurdity would follow, from supposing the existence of any other

Alternative Proof of III 13

(1) Let ○ *ADE* touch the ○ *ABC* *internally* at *A*, (See fig to III 11), then, except *A*, there can be no other pt of contact Find *F* the centre of the ○ *ABC* (III 1) Then *G* the centre of the ○ *ADE*, must be in *FA* (III 11) Take any point *E* in the ○ce of the ○ *ADE*, and join *FE* Since from *F*, (a point *within* or *without* the ○ *ADE*), *FA* and *FE* are drawn to the ○ce of the ○ *ADE*, (of which *FA* passes through the centre of ○ *ADE*) ∴ $FA > FE$ (III 7 and III 8, *cor*), but *FA* is the radius of ○ *ABC*, $FE <$ the radius, and hence *E* is *within* the ○ *ABC* So, every pt of the ○ce of the ○ *ADE* (except *A*), can be proved to be *within* the ○ *ABC* Hence, *A* is the *only* pt at which the ○s *touch* one another

(2) Let the ○s *ABC* and *ADE* touch one another *externally* at *A* (See fig to III 12), then, except *A*, there can be no other point of contact Find *F* the centre of the ○ *ABC* (III 1) Then *G* the centre of the ○ *ADE*, must be in *FA* produced (III 12) Take any point *D* on the ○ce of the ○ *ADE*, join *FD*, from *F*, (a pt *without* the ○ *ADE*), *FA* and *FD* are drawn to the ○ce of the ○ *ADE* (of which *FA* when produced, passes through *G* the centre of ○ *ADE*), $FD > FA$ (III 8) but *FA* is the radius of ○ *ABC*, *FD* is $>$ than the radius, and hence *D* is *without* the ○ *ABC* So every point of the ○ce of the ○ *ADE* (except *A*) can be proved to be *without* the ○ *ABC* Hence *A* is the *only* pt at which the ○s *touch* one another

If the line joining the centres of two ○s = the difference of their radii, they have *internal* contact, and if it be = the sum of their radii, they have *external* contact

Apply III 7 and 8, in proving the following *Cors* —

(1) If one ○ be contained *within* another without meeting it, the distance between their centres is $<$ than the difference of their radii

(2) If the distance between the centres, be $<$ than the difference between the radii, the *lesser* ○ will be contained within the *greater* without meeting it

(3) If two ○s lie each *without* the other, and do not meet the distance between the centres, is $>$ than the sum of the radii

(4) If the distance between the centres of two ○s, be $>$ than the sum of the radii, they lie each without the other and do not meet

PROPOSITION XV

To draw the Least chord, through a given pt. in a ○.

The *least* chord which can be drawn through a given point A in a ○, is the ⊥ to the diameter, which passes through A.

The *greatest* chord is the *diameter* (III 15) Through A draw any diameter DAB, and through A (the given pt), draw EAF ⊥ to DAB It is required to prove that EF is the *least* chord Through A draw any other chord GH, and from C the centre, drop a ⊥ CI on GH Now ∠ CIA = a rt ∠, and ∠ CAI is *acute* (I 17), ∴ $CA > CI$, and ∴ $GH > EF$ (III 15), and since the same is true of any other chord, it follows that EF is the *least chord*

N. B The *less* the ∠, a chord makes with the diameter through A—the *greater* the chord will be For, as ∠ HAB diminishes, the ⊥ CI will also diminish.

PROPOSITION XVI

III 16 is very important. From this, it follows that —

(1) If several ○s touch each other, either *internally* or *externally*, they have at their point of contact a *common tangent*, for the same straight line is ⊥ to that, which passes through their centre.

(2) Tangents through the extremities of the same diameter, are ∥*l*.

(3) Any linear magnitude, is capable of being infinitely divided Let BF be a tangent at B to the ○, whose centre is C Draw any line CI, meeting the ⊙ at O The line OI may be infinitely divided by describing ○s with centres C, D, E, F, *etc.* taken in BC (the diameter of the ⊙, whose centre is C) produced, touching the tangent BF at B.

N B To draw a tangent to a point on a ○, it is only necessary to draw a *diameter* through that point, and to draw a line ⊥ to it

PROPOSITION XVII

THE BEST METHOD OF DRAWING A TANGENT TO A ⊙ FROM A GIVEN POINT WITHOUT IT.

Join the *given point* and the *centre* of the ○; upon this line, describe a *semi*-⊙ cutting the given ○; then the line drawn from the given point to the *intersection*, will be a tangent. (See p 202, Text Book.)

III 17 is very important

The *Corollary* appended to III 17, is also **important**

N B It is obvious that, *two* tangents, and not more, can be drawn from the *external* point *A* for the ⊥ to *EA* through *D*, meets the ○ *GAF* in the points *G* and *F*, and no more ; each of these points, will determine a tangent

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PROPOSITION XVIII

III 18 may be **otherwise enunciated**, thus—All radii are *normals* to the ○ at the points, where they meet the ◡.

III 18 is the converse of III 16, ∵ a tangent to any pt in the ◡ce of a ○, is a st line at rt ∠s at the extremity of the diameter which meets the ◡ce in that pt

N B III 18 can be proved by the "*Method of limits*" (See p 230, Text Book)

PROPOSITION XIX

Alternative enunciation or III 19—Every normal to a ○ passes through the centre

Cor If two ○s *ABC* and *ADE* touch each other in any pt *A*, they have the same tangent at the pt. of contact

Through the pts of contact *A*, draw *FG* touching the ○ *ABC* (III 17), and through the same point, draw *BAD* at rt ∠s to *FG* (I 11) ; ∵ *FG* touches the ○ *ABC*, and *AB* is at rt ∠s to it, the centre of the ○ is in *BA* (III 19), and the st line *BD* passes through the centre of one of the ○s and the point of contact, it passes through the centre of the other ○ *ADE* (III 12), ∴ *AD* is a diameter of that ○, and *FG* is drawn at rt ∠s to the diameter *AD* (hyp), it touches the ◡ce of the ○ *ADE* (III 16, cor), the line *FG* is a *common tangent* to the ○s *ABC* and *ADE*

N B III 16, 18, 19 are *limiting cases* of III 1 Cor and III 3 *viz*, when the points in which the chord cuts the ○, become *consecutive*

III 16, 18, 19 are so related that, any two can be inferred from the third. The exact relation among these three propositions, can be best understood from the **Index**

PROPOSITION XX.

III. 20 consists of five separate cases *Euclid has only given two cases* He has not given the other *three* cases —

(1) When the ∠ at the centre, is on a side of the ∠ at the ⊙ce (Taking the fig of III 20, case 1), the ∠ *BEF* at the *centre* is on a side *AF* of the ∠*BAF* at the ⊙ce By applying I 5 and I 32, we know that ∠*BEF*=2 ∠*BAF*

(2) When the ∠ at the centre is a *straight* ∠ The proof is similar to case I

(3) For the proof of the case, when the ∠ at the centre is a *reflex* ∠ (See Text Book p 165)

N B. In III, 20 case 1, it is **assumed** that, if there be *four* magnitudes, such that the first is *double* of the second, and the third double of the fourth—then the first and third together shall be double of the second and fourth together, also in (III 20 case 2nd), it is *assumed* that, if one magnitude be double of another, and a part taken from the first be double of a part taken from the second, —the remainder of the first, shall be double the remainder of the second

Def If in the ⊙ce of a ⊙, two pts *B* and *C* be taken, and if these two pts be joined to the centre *E* of any ◡ *ABC*, then ∠*BEC* is called *the ∠ at the centre*, and if two other lines be drawn from *B* and *C* to any point *A* on the ⊙ce, then ∠*BAC is called the ∠ at the ⊙ce*

PROPOSITION XXI

III. 21 is very important

ALTERNATIVE PROOF OF III 21

In the arc *BCD*, take any pt *C*, and join *BC* and *DC* Then the quadrilateral *ABCD* is contained in the ⊙, its opposite ∠s *A* and *C* are together=2rt ∠s (III 22), and ∵ the quadrilateral *EBCD* is contained in the ⊙, its opposite ∠s *E* and *C* are together=2rt. ∠s (III 22); ∠s *A* and *C*=∠s *E* and *C* From these equals, take the common ∠ *C*, ∴ ∠*A*=∠*E*

ALTERNATIVE PROOF OF CASE 2ND, III. 21.

Here the segment *BAED* is not *greater* than a semi=⊙ Join *AE* Let *AD* and *BE*, cut one another at *X*. In △s *ABX* and

EDX, $\angle AXB = \angle EXD$ (I 15), and $\angle ABX = \angle EDX$ (III 21, case 1), $\therefore \angle BAX = \angle DEX$ (I 32) *i e* $\angle BAD = \angle BED$

N. B If the term '*angle*' had been *extended* by Euclid, as it has been in *Modern Science* to $\angle$s *greater than two rt* $\angle$s, no other subdivision of this demonstration into cases, would be necessary (See p 187, Text Book)

The converse of III 21, is very important (See p 187, Text Book)

PROPOSITION XXII

III 22 is very important. It can be **otherwise enunciated** thus —

(1) The opposite $\angle$s of *cyclic* quadrilateral, are *supplementary* (2) The sum of one pair of opposite $\angle$s of *a cyclic* quadrilateral = sum of the other pair

Alternative Proof of III. 22

See figure of III 22, p 189, Text Book

Let F be the centre of the ⊙ $ABCD$ Join FD, FC, FB and FA Since $FD = FC = FB = FA$ (being radii of the same ⊙), Δs FDC, FCB, FBA, and FAD are each isosceles, $\therefore$ in ΔFDC, $\angle FDC = \angle FCD$, and in ΔFBC, $\angle FCB = \angle FBC$, $\therefore \angle DCB = \angle FDC + \angle FBC$ So in ΔFDA, $\angle FDA = \angle FAD$, in ΔFAB, $\angle FAB = \angle FBA$, $\therefore \angle DAB = \angle FDA + \angle FBA$ Hence $\angle DCB + \angle DAB = (\angle FDC + \angle FBC) + (\angle FDA + \angle FBA) = (\angle FDC + \angle FDA) + (\angle FBC + \angle FBA) = \angle CDA + \angle CBA = 2$ rt $\angle$s

For, sum of the 4 $\angle$s of a quadrilateral $= 4$ rt $\angle$s (I 32 cor) And of the 4 $\angle$s, if the sum of one pair of $\angle$s = the sum of the other pair of $\angle$s, then each sum $= 2$ rt $\angle$s

Another proof of III 22, can be obtained by applying III 20 (See Text, p 189)

The converse of III 22, is very important For the converse of III 22, (See Text, p 189)

Cor If one side AB of a quadrilateral fig $ABCD$ contained within a ⊙ be produced, the *external* $\angle CBE = \angle ADC$, opposite to the internal adjacent $\angle$

For, the opposite $\angle$s ADC and $ABC = 2$ rt $\angle$s (III. 22), and $\angle$s ABC and $CBE = 2$ rt $\angle$s (I 13), $\therefore \angle$s ADC and $ABC = \angle$s ABC and CBE, and taking from each, the common $\angle ABC$, $\angle ADC = \angle CBE$.

PROPOSITIONS XXIII, XXIV, and XXV

It is evident from III. 23, that, of two circular segments on the same base, the larger is that which contains the smaller $\angle$

From III. 24, it follows that, "similar segments having equal chords, have also equal arcs"

Since, two $\odot$s must coincide in every part, agree in more than two points,—it follows that, "similar segments having equal chords, are parts of equal $\odot$s"

From III. 24, it follows, that if the radii of two $\odot$s are equal, the $\odot$s themselves are equal, and their $\odot$ces equal

Sectors whose radii and $\angle$s are equal, are themselves equal.

Apply I. 4, III 20 and III. 22

III 25 may be otherwise enunciated thus —(1) To find the centre of a segment of a $\odot$ (2) Or a segment of a $\odot$ being given, to describe the $\odot$, of which it is a segment.

PROPOSITIONS XXVI, XXVII, XXVIII, AND XXIX.

It is better to insert the words "*or on the same* $\odot$" after "in equal $\odot$s" in the enunciation of each of the *four* propositions. III 27 is the converse of III. 26

N B The relation existing between III. 26, 27, 28, 29, be best understood by separating the *hypothesis* from the *consequence* See the Index

	Hypothesis	*Consequence*
In III. 26, In equal $\odot$s or in the same $\odot$	Central or circumferential $\angle$s being equal	*Arcs* shall be equal.
" " 27 " "	*Arcs* being equal	Central or circumferential $\angle$s shall be equal
" " 28	*Chords* being equal.	*Arcs* shall be equal
" " 29 .	*Arcs* being equal	*Chords* shall be equal

N B If in equal $\odot$s, or in the same $\odot$, either of the *five* pairs, (*arcs, chords, $\angle$s at the centre, $\angle$s at the* $\odot$*ce*, or *sectors*)—are equal, the other *four* pairs are equal

In the same $\odot$, equal, central or circumferential $\angle$s, stand upon equal arcs.

Let $MXYN$ be a ○, R its centre, ∠XRY (at the centre)= ∠MRN (at the centre), and ∠XPY (at the ○ce)= ∠MQN (at the ○ce) Then it is required to prove that, arc XY shall be=arc MN Join MN and XY In △s MRN and XRY, MR, RN=XR, RY and ∠XRY=∠MRN, ∴ MN=XY (I 4) Since, ∠XPY= ∠MQN, ∴ segments MQN and XPY are *similar*, and since they are on equal lines MN and XY, ∴ segment XPY=segment MQN (III 24) Hence, ○ $MXYN$—segment XPY=○ $MXYN$— segment MQN, the remainders are equal, hence arc XY =arc MN

N B These *four* propositions, can be *directly* proved by the method of *Superposition* (See p 189, Text)

Important Results

(1) If the opposite ∠s of a quadrilateral in a ○, be equal—the diagonal opposite to them, must be a diameter, and since in this case, the ∠s are both *right* ∠s, it follows that a segment containing a rt ∠, is a semi-○

(2) If one central or circumferential ∠ in the same or equal ○s, be > than another—the arc on which the one stands, will be >than that on which the other stands Hence, if, of two opposite ∠s of a quadrilateral inscribed in a ○, one be *acute* and the other *obtuse*—the arc on which the former stands will be <, and the latter >than a semi ○, hence, the segment which contains an acute ∠, is > and that which contains an obtuse ∠, is < than a semi-○

For the converse of III 31, (See Notes on III 31)

(3) Supplemental circumferential ∠s in the same or equal ○s, stand on arcs, whose sum=a whole ○ce

(4) Diameters intersecting at rt ∠s, divide the ○ce into *four* equal arcs

(5) Any number of central ∠s in the same or equal ○s, whose sum=*four* rt ∠s stand on arcs whose sum=a whole ○ce

(6) Any number of circumferential ∠s, in the same or equal ○s, whose sum=2rt ∠s—stand on arcs, whose sum=a whole ○ce

(7) The sum of the central ∠s subtended by arcs whose sum =an entire ○ce=4 rt ∠s The sum of the *circumferential* ∠s subtended by the same arcs=2 rt ∠s

(8) A *quadrant* subtends a rt ∠ at the centre, and a *semicircle* subtends a rt ∠ at the ○ce

PROPOSITION XXX

From III 30, it is evident that the st. line drawn through the centre of a ○, to bisect the chord—bisects the arc, and if it bisects the arc, it also bisects the chord

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PROPOSITION XXXI

III. 31 is very important

Alternative Proofs

A. (Part II) (1) If ∠ *ABC* in a segment > than a semi○ is *acute*

Draw *AD*, a diameter of the ○, and draw the lines *CD*, *CA*. Since ∠ *ACD* in a semi○, is a rt. ∠ (Part 1. III. 31), ∠*ADC* is *acute* (I 32), but ∠ *ADC* = ∠ *ABC*, being in the same segment *ABDC* (III 21); ∴ ∠ *ABC* is *acute*

(2) Otherwise —(In the above figure), join *DB* instead of *DC*, ∴ ∠*ABD* = a rt ∠, ∴ ∠*ABC* is *acute*

(Part II) (1) If ∠ *ABC* in a segment < than a semi○ is *obtuse*. Take in the opposite ○ce, any pt *D*, and draw *DA* and *DC*, ∵ in the quadl. *ABCD*, the opposite ∠s *B* and *D* = 2 rt. ∠s (III 22), but, ∠ *D* < a rt ∠ (Part I, III 31), ∠*ABC* must be *obtuse*

(2) Otherwise —Draw the diameter *AD*, and draw *BD*; ∠*ABD* is a rt ∠, ∴ ∠*ABC* is *obtuse*

B. III 31 can be deduced from III. 20, (See p 202, Text)

Converse of III. 31

If an ∠ in a segment is <, = or > than a rt ∠, the segment is >, = or < than a semi○ (See Notes on Prop 26-29, B III)

N B (1) III 17, can be deduced from III. 31. (See notes on III 17)

(2) From III. 31, we can draw a line through the extremity of the given line ⊥ to the same

Let *AB* the given line. From *C* (any point without *AB*), and with *CB* as a radius—describe a ○ cutting *AB* in *D*, join *DC*, and produce the line to cut the ○ce of the ○ in *E*, draw *BE*, and it will be the ⊥ required. For ∠ *DBE* being in the semi○ *EBD*, is a rt. ∠ (III 31)

(3) From III 31, it is evident that, *the middle point of the hypotenuse of a rt. ∠d △, is equidistant from the three angular points.*

PROPOSITION XXXII

Alternative proof of III 32

Let EF touch the ○ ABC at B Let BD be a chord drawn from B, the pt. of contact Then (i) ∠ DBF= ∠ in the *alternate segment* BAD, (ii) the ∠ DBE= ∠ in the *alternate segment* BCD.

From B draw BA ⊥ to EF Take any point M in the semi⊙ AMB Join AM, MB and MD ∠ AMB=a rt ∠, ∵ BA is a diameter, (III 31) and ∠ ABF=a rt ∠ (const), ∠AMB= ∠ABF (each being a rt ∠) Now ∠AMD= ∠ABD (being in the same segment) (III 21). ∠AMB—∠AMD or ∠DMB= ∠ABF—∠ABD or ∠ DBF

Take any point C in the arc BD Join AC, ∠ ACB=a rt ∠, ∴ BA is a diameter (III 31), ∠ ABE=a rt ∠ (const) ∠ DCA=∠DBA) (III 21), ∠ ABE+ ∠DBA=∠ACB+∠ DCA or ∠ DBE= ∠ DCB in the *alternate segment* BCD

N B Hence III 32 is a particular case of III 21

Converse of III 32

If a st line meet a ○, and from the point of meeting a st line be drawn cutting the ○, and the ∠ between the two st lines be= to the ∠ in the *alternate segment* of the ○, the st line which meets the ○, shall *touch* the ○

In the figure of III. 32, if possible, let EF which meets the ○ ABC at B, does not touch it Through B draw another line PB to touch the ○

Then ∠ PBD=∠BAD (III 32) But ∠ FBD=∠ BAD (hyp), ∴ ∠PBD=∠FBD, the part=the whole, which is impossible, EF touches the ○ (See Ex I p 204 Text)

Important results

(1) To draw a tangent from any point on the ○ce of a ○ *without finding its centre*

Let XYZ be a ○, X and Y are two points on its ○ce With Y as centre and YX as radius, describe a ○ cutting the ○ XYZ at Z, and XY produced at P Make arc PQ=arc PZ. and join XQ Then XQ shall be the tangent to the given ⊙ XYZ. Join XZ.

∠PXZ=∠PXQ (III 27), ∠ PXZ or ∠YXZ=∠YZX (I 5); ∠PXQ=∠YZX, ∴ XQ is a tangent to the ○ XYZ (Converse of III 32)

(2) If several ⊙s touch each other, either *internally* or *externally*, any st line passing through the point of contact, will cut off similar segments from them For since they have a *common tangent*, the ∠s in all the segments = ∠ under the line drawn and the common tangent

(3) If several ○s touch each other *internally* or *externally*, and two st lines be drawn through the pt of contact P cutting each of them at A and B, the lines AB will be ||| for by (1) the alternate ∠s PAB are equal

(4) Tangents through the extremities of the same chord, make equal ∠s with it on the same side For each ∠ = ∠ in the *alternate segment*

(5) The chord, joining the pts of contact of ||| tangents, is a *diameter*. For the ∠s of the same side are equal (3), and *supplemental* (I 29), and are . *right* ∠s , the chord is a diameter,(III 19)

PROPOSITION XXXIII

From III 33, we derive the solution of another proposition — *Given the base, and vertical ∠ of a △, to find the locus of the vertex* (See p 206 Text)

This *problem* is useful in the solution of all problems relating to the determination of a △, where two of the three data are a *side* and the ∠ opposite to it In such cases, having constructed on the given side, the segment which contains the opposite ∠, all that remains to be determined is the pt in this segment, where the vertex is placed The third *datum* ought to be sufficient to determine this Thus, for example, if the 3rd datum be the ⊥ from the vertex on the given side, the place of the vertex may be determined by drawing any line ⊥ to the given side, and taking a part on it from the side = the given ⊥ A ||| line to the side through the extremity of this, will intersect the ○ in two points, either of which will serve for the vertex

If the base, vertical ∠, and the ⊥ from the extremity of the base, on the opposite side be given, to find the △.

On the given side AB describe a segment ACB containing the given ∠ (III. 33) And describe a semi ○ ADB It is evident that the vertex of the △ must be in the *former*, and the point where the ⊥ meets the side in the *latter* Inflect AD = the ⊥, and draw BD to meet the *first* segment at D, the △ ABC is the △ required

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PROPOSITION XXXV.

III 35 is very important

$AE \cdot EB + GE^2 = GA^2$ (II 5) (for AB is divided *equally* at G, and *unequally* at E) $AE \cdot EB + (GE^2 + GF^2) = GA^2 + GF^2$,

(1) $AE \cdot EB + EF^2 = AF^2$ (I 47) $=$ (Radius)^2

And $CE \cdot ED + EH^2 = HD^2$ (II 5), (for CD is divided *equally* at H, and *unequally* at E)

(2) $CE \cdot ED + (EH^2 + HF^2) = HD^2 + 2HF$ or $CE \cdot ED + EF^2 = DF^2$ (I 47) $=$ (Radius)^2

From (1) and (2), $AE \cdot EB + EF^2 = CE. ED + EF^2$, hence $AE \cdot EB = CE \cdot ED$

Different cases of III 35 (See p 209 Text).

(1) *When the given chords AB and CD both pass through the centre F* –

Then it is evident, that, they are both bisected in the pt of intersection , the rect under their segments are the squares on their *halves*, and are equal

(2) *When one chord AB passes through the centre F, and cuts the other CD at rt $\angle$s in E*

Join FC , AB passes through the centre F and cuts CD which does not pass through the centre at rt $\angle$s, . AB bisects CD at E (III 3)

$AE \cdot EB + FE^2 = FB^2$ (II 5) $= FC^2 = FE^2 + EC^2$, taking away FE^2, we have $EA \cdot EB = EC^2 = CE \cdot ED$, (for $CE = ED$)

(3) *When one chord AB passes through the centre F and cuts the other CD obliquely at E*

Drop $FH \perp$ to CD from F, join FD , CD is cut at rt $\angle$s by FH which passes through the centre F Since CD is bisected in F (III 3) and divided *unequally at E*

$CE \cdot ED + EH^2 = HD^2$ (II 5) , $CE \cdot ED + (EH^2 + FH^2) = HD^2 + FH^2$, $CE \cdot ED + EF^2 = FD^2$ (I 47) $=$ (Radius)^2 (1)

$\because AB$ is bisected at F, and divided unequally at E, $AE \cdot EB + EF^2$ (II 5) $=$ (Radius)^2 (2) From (1) and (2), we have $AE \cdot EB + EF^2 = CE \cdot ED + EF^2$, hence rejecting EF^2, we have $AE \cdot EB = CE.ED$

(4) If two chords AB and CD of a $\bigcirc ABDC$, when produced, cut one another at E, the rectangle contained by the external segments of the one, shall be $=$ to the rectangles contained by the ex-

ternal segments of the other; or it is required to prove $AE\ EB = CE\ ED$ (See Ex 7, p 209 Text)

Find F the centre of the ○, FG and FH are drawn from the centre, ⊥ to chords AB and CD respectively, AB is bisected at G and CD is bisected at H (III 3)

∴ AB is bisected at G, and produced to E, $AE\ EB + GB^2 = GE^2$ (II 6), or $AE\ EB + GB^2 + GF^2 = GE^2 + GF^2$ or (1) $AE.EB + FB^2 = FE^2$ (I 47) Similarly (2) $CE\ ED + FD^2 = FE^2$, $AE\ EB + FB^2 = CE\ ED + FD^2$, but $FB^2 = FD^2$ for FB = a radius $= FD$. Rejecting these equals, we have $AE\ EB = CE\ ED$

N B Case 4 is given as Cor to III. 36

Converse of III 35

If two st lines AB, CD cut one another at E, so that the rectangle contained by the segments of the one (AE and EB) is = to the rectangle contained by the segments of the other (CE and ED), the *four* extremities A, B, C, D of the two st lines, are on the ○ce of a ○ (*See Ex 1 p 200 Text*)

Since a ○ can be described through any three points which are *not* in the same st line, describe a ○ through A, C, B If the ○ does not pass through the *4th point* D let it cut CD or CD produced at X Then $AE\ EB = CE\ EX$ (III 35) But $AE\ EB = CE\ ED$ (Hyp), $CE\ EX = CE\ ED$, $EX = ED$, which is impossible, the ○ which passes through A, B, C must pass through D

PROPOSITIONS XXXVI, and XXXVII

Symbolical form of the proof of III 36.

$AD\ DC + FC^2 = FD^2$ (II 6) for AC is bisected at F, and produced to D, $AD\ DC + (FC^2 + FE^2) = FD^2 + AE^2$ (Adding EF^2 to each), or $AD\ DC + EC^2 = ED^2$ (I 47) But $ED^2 = EB^2 + BD^2$ (I 47), $AD\ DC + EC^2 = EB^2 + BD^2$ (Ax 1), and rejecting the equals EC^2 and EB^2, we have $AD\ DC = BD^2$

When the secant AD passes through E, the centre of the ○ BD.

(Proof) $AD\ DC + EC^2 = ED^2$ (II 6) $= EB^2 + BD^2$ (I 47) and rejecting the equals EC^2 and EB^2 we have $AD\ DC = BD^2$

N B III 36 and the *Corollary* given in p 211, are very important

III 37 is the Converse of III. 36

QUESTIONS ON BOOK THIRD

161 Define —Secant, tangent, sector, major arc, angle *in* a segment, angle *of* a segment, concentric circles, similar segments

of circles, concavity, convexity, concyclic points, cyclic figure, angle in the alternate segment of a circle, direct and transverse common tangents, equal circles, orthogonal circles, polar of a point, radical axis and radical centre, co-axal circles, orthocentre, and pedal or orthocentric triangle

162-76 Give the *Alternative proofs* of the following propositions Book III —

1st, 2nd (case 2), 3rd (case 2), 5th, 6th, 10th (2nd proof), 11th, 15th, 17th, 20th, 21st, 22nd, 31st, 32nd and 35th

177-85 State and prove the *Converse* of the following propositions, Book III —

Cor to 1st, 3rd, 11th, 12th, 21st 22nd, 31st, 32nd and 35th

186 Give a list of all the *Converse Propositions*, Book III

187-91 Give the *Different Cases* of —

III 10, 20th, 31st, 35th and 36th

192 What is the *Subject Matter* of Book III ?

193-95 Give the *Direct* Proofs of —

III 11th, 12th, 13th, and 16th

196-98 What is the difference between —a chord and a secant, a sector and a segment of a ○? When does a secant become a tangent? When does a *sector* become a segment?

199-200 How many intersections can a *line* and a ○ have? How many *points of intersections* can two ○s have?

201 If two ○s *touch*, they cannot have any other common point Prove this

202-10 Give the *Alternative Enunciations* of the following propositions Book III. —

5th, 7th, 8th, 11th, 12th, 18th, 19th, 22nd and 25th

211-12 Include III (5 and 6), (11 and 12), under *one enunciation*

213 What axiom is assumed in proving III 20?

214. What does the line become, when its points of intersections with a ○, become *consecutive*?

215-17 State the relations between —III 1 cor and III 3, III (16, 18, 19), III (26, 27, 28, 29)

218-21 What propositions are limiting cases of (III. 16, 18, 19)? How many common tangents can two ○s have?

222-23 Give the symbolical form of proofs of III 35 and 36

224-25 Deduce III 9 as a corollary from III 7 (case 4), and III 31 from III 20

226. Draw a tangent to a ⊙ from any point *in* the ⊙ without finding its centre

227 Prove that in the *same* ⊙, equal, central or circumferential ∠s, stand upon equal arcs

228 To draw the *least* chord through a given point in a ⊙

229-31 State the assumptions in —III 1st, 7th, 20th

BOOK FOURTH

Subject matter —The subject-matter of Book IV is the inscription and circumscription of △s and of regular polygons in and about ⊙s

N B (1) Euclid has not given any instance of the inscription or circumscription of rectilineal figures *in* and *about* other rectilineal figures

(2) **This book consists entirely of Problems**

(3) Out of 16 propositions, 4 relate to △s, 4 to *squares*, 4 to *pentagons*. besides these. there are 4 other *miscellaneous* propositions

(4) **Propositions 3rd, 4th, 5th, 10th, 11th, 12th, 15th, and 16th are important**

PROPOSITION I

The restriction that a "chord = a st line not > than the diameter", in the enunciation of IV I. is important, for we cannot place in a ⊙ a st. line > than the diameter (the *greatest* chord) III. 15

PROPOSITION II

Euclid has omitted here to state, that the st. lines *AC* and *AB* must be drawn on the *same* side of the tangent as the ⊙, and has *assumed* that these lines will *cut* the ⊙ce

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PROPOSITION III

Euclid here omits to prove that the lines (*MN*, *NL* and *LM*) which touch the ⊙ at *B*, *C*, *A*, must necessarily *meet* when produced. This may be proved thus by joining *AB* .—∠*KAM*+

$\angle KBM = 2$ rt $\angle$s (III 18), $\therefore$ $\angle BAM + \angle ABM$ are $<$ 2rt $\angle$s, $\therefore$ AM and BM must meet, when produced (Ax 12) So, it may be shewn that AL, and CL, as also CN and BN meet one another

Alternative Proofs of IV. 3.

(1) Let BAC be the given ○, X its centre, and DEF the given △ Draw BXG any diameter Make $\angle GXA = \angle E$, $\angle GXC = \angle F$, and at A, B, C draw tangents intersecting at L, M, N Then LMN is the reqd △ For $\angle XAM = \angle XBM =$ a rt $\angle$, (III 18), the points A, X, B, M are *concyclic*, $\angle GXA = \angle M$ (III 22 cor or Ex 3 p 188 Text), $\therefore$ $\angle M = \angle E$ Also $\angle N = \angle GXC = \angle F$, $\therefore$ $\angle L = \angle D$ (I 32 cor)

(2) Take any point B on the ○ce of the ○ ABC, draw a tangent XBY through B (points X and Y are on opposite sides of B) At X make $\angle YXZ = \angle E$, and at Y make $\angle XYZ = \angle F$ Supposing that XZ and YZ do not *touch* the ○ ABC, draw $\perp$s to XZ and YZ from centre K Let these $\perp$s meet the ○, (produced if necessary) at A and C Through A and C draw tangents LAM and LCN (meeting one another at L) and (meeting XY or XY produced) at M and N respectively Then LMN is the △ required

(*Proof*) Since ZX and LM are both $\perp$ to AK, ZX is $\|$l to LM, $\therefore$ $\angle LMN = \angle YXZ$ (I 29) $= \angle E$ For similar reason, ZY and LN are $\|$l, $\therefore$ $\angle LNM = \angle XYZ$ (I 29) $= \angle F$, $\therefore$ $\angle MLN = \angle D$ (I 32 cor)

PROPOSITION IV

It is assumed here, that the two bisectors of $\angle$s B and C will meet at the *same* point This may be proved thus, $\angle ABC + \angle ACB$ are $<$ 2 rt $\angle$s (I 17), $\frac{1}{2}(\angle ABC + \angle ACB)$ are much more $<$ 2 rt $\angle$s, BI and CI will meet at I, (Ax 12)

IV 4 is a *particular case* of the more *general problem* —To describe a ○ touching three given st lines

1 If the three given lines be $\|$l to one another, the problem is *impossible*, since no ○ touching two of them could touch the 3rd

2 If two of the lines AG, BH be $\|$l, and the third AB intersect them

Draw the lines AD and BD bisecting the $\angle$s A and B These will intersect, since they make $\angle$s with AB which are together $<$ than 2rt $\angle$s Let them meet at D, $\perp$s DE, DG, and DH to the

three given lines from D, are equal. This may be proved as in IV 4 Hence D is the centre, and DF the radius of the ○

There are two ⊙s which touch the given st lines

Let the three given st lines intersect, so as to form a △

In this case, the ⊙ is determined as in IV 4 But this is not the only ⊙ which may be drawn touching the given st lines Draw the lines CD and AD bisecting the external ∠s at A and C These will meet at D, and ⊥s DE, DF, DG on the given st lines from this point, are equal Hence D is the centre, and DF the radius of a ○ touching the three given st. lines. The demonstration of this is exactly the same as that of IV. 4 (See the construction of *Escribed* ○s)

In the same manner, two other ○s may be described touching the given st lines

Thus, if three st lines intersect, so as to form a △, *four* different ⊙s may be described each touching them all

N B—By this case, it appears that, the bisector of any internal ∠ of a △, and those of the remaining external ∠s, intersect at the same pt

4 If the three given lines intersect at the same point, no ○ can be described touching them all

It is evident that the problem "*To describe a* ○, *touching two given st lines*" is **indeterminate** We can, however, in this case, determine the **locus** of its centre.

1 *If the two st lines be* ||*l*.

Draw the line AB intersecting them ⊥ly, and bisect it at C, and through C, draw DF |||l to the given st. lines This will be the *locus* of the centres For, if any other ⊥ FG be drawn, a ○ described on it as the diameter, will touch the given lines

2 *If the given lines intersect*

Draw the lines AB and CD bisecting the ∠s under the given lines. These lines will be the *locus* of the centres

The *results* given in p 255, Text Book—are *very important. To draw an escribed* ⊙ of a given △ (See p 255 Text).

PROPOSITION V

IV 5 is, in fact, the same as *to describe a* ⊙ *through three given points, which are not in the same st line.*

N B—A ⊥ from S will evidently bisect BC, and the ⊥s from the middle pts of the sides of a △, have a common pt of

intersection (See p 103 Ex 1 Text), and this point is the centre of the circumscribed ○

It is assumed in the demonstration of IV 5, that the ⊥s through D and E will intersect, if produced This may be proved by drawing the st line joining D and E The ⊥s evidently make with this line, ∠s which are together < than 2 rt ∠s

The Notes on pp 256 257, Text, are important

N B If the △ be equilateral, the centre of the inscribed ○, is equidistant from the angular points of the △ The centres of the ○s *inscribed in*, and *circumscribed about* an equilateral △ coincide, and the radius of one = *double* the radius of the other

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PROPOSITIONS VI and VII

The inscribed square is = twice the square on the radius, or = half the square on the diameter

$AB^2 = AE^2 + BE^2 = 2AE^2$, and $AC^2 = AB^2 + BC^2 = 2AB^2$, ∴ $AB^2 = AC^2$

The circumscribed square = to the square on the diameter, and is = twice the inscribed square, and = 4 (radius)^2

From IV 7, Circumscribed square = (Diameter)^2 = 2 inscribed square = 4 (radius)^2, see above

A square is the only rt ∠d □m, which can be *circumscribed* about a ○, but both a *rectangle* and a *square* may be inscribed in a ○

PROPOSITION X

1 The △ BCD is also isosceles and has each of the ∠s at its base (∠ B and ∠ BDC), *double* of the vertical ∠ BDC (See Proof of IV 10)

2 ∠ A (in IV 10) = ⅕ of 2 rt ∠s, for ∠A + ∠B + ∠ADC = 2 rt ∠s (I 32), or ∠A + 2∠A + 2∠A = 2 rt ∠s (IV 10), hence 5 ∠A = 2rt ∠s, ∴ ∠A = ⅕ of 2rt ∠s or = 36^0

3 The △ ACD is isosceles, and each of the ∠s at the base (∠A and ∠ADC) = ⅓ of the vertical ∠ ACD, for ∠ACD = ∠B + ∠BDC (I 32) = 2 ∠A + ∠A = 3∠A, and ∠ADC = ∠A, ∴ ∠A or ∠ADC = ⅓ of ∠ACD, and since ∠A = 36^0, ∴ ∠ACD = 108^0.

Alternative Proofs of IV. 10

1 *Without constructing the larger* ⨀

Divide AB in internal *medial section* in C, so that $BA \cdot BC = AC^2$ (II 11) Describe a Δ ADB on AB, making $AD = AB$ and $DB = CA$ (I. 22) Then proceed as in IV 10.

2 *Without constructing the smaller* ⨀

[*See after* "*join CD.*] From D, draw $DX \perp$ to BC. Now ∵ $\angle B$ is *acute*, $AD^2 = AB^2 + BD^2 - 2\,AB \cdot BX$ (II 13) Now $AD = AB$, ∴ $AD^2 = AB^2$; ∴ $DB^2 = 2\,AB \cdot BX$, but $BD^2 = AB \cdot BC$, for $BA \cdot BC = AC^2 = BD^2$ (constr.) Thus $AB \cdot BC = 2\,AB \cdot BX$, ∴ $BC = 2\,BX$; hence Δ DBC and Δ CAD are isosceles Then proceed as in IV 10

PROPOSITION XI.

Each diagonal of a regular pentagon is ∥ *to the side, with which it is not conterminous*, for $\angle BAC = \angle ACE$ (III 27); ∴ AB is ∥ to CE

In IV 11, the *pentagon* is inscribed in the ◯, by the aid of an isosceles Δ whose base $\angle$s are each $= 2$ vertical $\angle$, so any other equilateral fig of any number of sides may be inscribed in a ◯ by the aid of an isosceles Δ, in which (each of the base $\angle$s) *is to* (its vertical $\angle$) *as* (half the number of its sides—$\frac{1}{2}$) is to *unity*, thus, a square may be inscribed by the aid of an isosceles Δ having the ratio between each of its base $\angle$s and vertical $\angle$, as $(\frac{4}{2}-\frac{1}{2})$ or $1\frac{1}{2} : 1$, a *pentagon*, as $(\frac{5}{2}-\frac{1}{2})$ or $2 : 1$, a *hexagon*, as $(\frac{6}{2}-\frac{1}{2})$ or $\frac{5}{2} : 1$, and so on

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PROPOSITION XV

Every equilateral figure inscribed in a ◯, *must be equiangular*, for its $\angle$s are contained in *equal arcs*, and ∴ stand on equal arcs.

The side of the *regular hexagon* is $=$ to the radius of its circumscribing ◯, and *its area is $=$ six times that of an equilateral* Δ constructed on the radius of this ◯

If any three alternate $\angle$s, A, C, E of the *hexagon* be joined by st lines, they will form the inscribed equilateral Δ.

QUESTIONS ON BOOK IV

232 What is the *Subject-matter* of B IV ?

233 Define —In centre, circum-centre, *escribed* ⊙, *nine points* ◯, *reciprocal* properties

234-35 When is one rectilineal figure said to be inscribed in another ? When circumscribed ?

236-37 When is a ◯ said to be inscribed in a rectilineal figure ? When circumscribed about it

238 Give instances of *reciprocal* propositions in B IV

239 Name the figure that can be *inscribed in*, and *circumscribed about* a ◯, by means of B IV.

240 What *three regular figures* can be used, in filling up *the space round a point* ?

241-42 Give the *Alternative Proofs of* —IV 3, IV 10

243 What relation subsists between the *square inscribed in*, and the *square described about* the same ◯

244 Shew that in the fig of IV 10, there are two △s possessing the required property

245 Shew that in fig of IV 10, there is an isosceles △ whose equal ∠s are each = one third of the third ∠

246 In the construction of Euc IV 3, Euclid has omitted to shew that the tangents drawn through the points *A* and *B*, will meet in some point *M* How may this be shown ?

247 Shew that if the point of intersection of the ◯s in Euclid's figure of IV 10 be joined with the vertex of the △, another △ will be formed equiangular and equal to the former

248 What regular figures may be inscribed in a ◯ by the help of Euc IV 10 ?

249 The difference of the squares described on the st lines joining the extremities of the base of the constructed △ in the figure of Euc IV 10, with the other point of intersection of the ◯s is equal to the square on the side of the △

250 *(1)* Shew that the area of an *equilateral* △ inscribed in a ◯ is $=\frac{1}{2}$ of a *regular hexagon* inscribed in the same ◯

(2) The *regular hexagon inscribed* in a ◯ is $=\frac{3}{4}$ of the *regular circumscribed hexagon*,

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END OF PART I.

PART II.

EXERCISES IN BOOK I.

1 *Find a point which is equidistant from two given points*

Let X and Y be the two points Join XY, and on it describe an equilateral $\triangle$ XYZ (I. 1) Z is the point required

2. *Draw a figure for the case in Proposition I 2, in which the given point coincides with B*

With centre B and radius BC, describe a ⊙ CDE and from B draw any line BD to meet the ○ce in D Then BD = BC

3 On a given straight line, describe an isosceles $\triangle$, having each of the equal sides = a given straight line.

Let AB be the given straight line, on which the isosceles $\triangle$ is to be described, and let X be the given straight line to which equal sides of the $\triangle$ are to be equal

In AB or AB produced, take AE = X In BA, or BA produced, take BF = X With centre A and radius AE, describe a ○. With centre B and radius BF, describe another ⊙ Let the two ⊙s meet in G, join AG and BG Then . AG = AE ∴ AG = X and BG = BF, ∴ BG = X AGB is the required isosceles $\triangle$.

N B—The length of X must be greater than half of AB (the given base), otherwise the two ○s will not cut each other

4 In the figure of I 2, if the diameter of the smaller ○ is the radius of the larger, shew where the given point and the vertex of the constructed $\triangle$ will be situated.

The given point and the vertex of the constructed $\triangle$ will be *evidently* on the ○ce of the smaller ⊙

5 (*a*) *If the difference of two straight lines be added to the sum of two straight lines, the result will be double of the greater straight line. And* (*b*) *If the difference of two straight lines be taken away from the sum of two straight lines, the result will be double of the less straight line*

(*a*) Let XY and YZ be the given straight lines of which XY is the greater; cut off XS = YZ. then XZ = XY + YZ and

SY=XY−XS or XY−YZ, ∴ SY + XZ = (XY−YZ)+(XY +YZ)=2XY

(*b*) Taking the figure of 5 (*a*), we have XZ−SY=(XY+YZ) −(XY −YZ)=2YZ

6 Describe an isosceles △ upon a given base and having each of the equal sides double of the base, without using any proposition of the Elements, subsequent to I 3 What condition must be fulfilled with regard to the magnitude of each of the equal sides *(Cam Ex Pap 1849)*

Let AB be the given base, produce AB both ways, cut off BE, AD, each=2AB (I 3), with centre A, and radius AD, describe a ○, and with centre B and radius BE, describe another ○, cutting the former in G Join GA, GB Then AG, BG are =AD, BE respectively (I def 11), but AD, BE are each= 2AB (constr), ∴ AG, BG are each=2 AB; and ∴ ABG is the isosceles △ required

N B—Each of the equal sides must be greater than half the base

7 *A line that bisects the vertical angle of an isosceles △, also bisects the base ⊥ly*

Let △ CAB be isosceles, and let CD bisect ∠ ACB In △s ACD, BCD, AC=BC, CD is common, ∠ACD=∠BCD, ∴ AD=BD, ∠ADC = ∠ BDC (I 4), CD is ⊥ to AB (I 4).

8 If two straight lines bisect each other at rt ∠s, any point in either of them, is equidistant from the extremeties of the other

Let AB, CD bisect each other at rt. ∠s at M Any point C in CD, shall be equidistant from A and B Join AC, BC, then, in the △s AMC and BMC, AM=BM, MC is common, and the ∠s at M are rt. ∠s, AC=BC (I 4) The same may be proved of lines drawn from any point in AB

9 The squares described on two equal straight lines are equal

Let ABCD, EFGH be squares described on the equal straight lines AB,EF For if sq ABCD be applied to sq EFGH, so that A falls on E, and so that AB falls on EF, then B will coincide with F, ∵ AB=EF (Hyp) And AB coincides with EF, and ∠ B = ∠F, (I Ax 10), BC will fall on FG And ∵ BC =FG, C will coincide with G Again, ∵ BC coincides with FG, and ∠C = ∠G, ∴ CD will fall on GH And ∵ CD=GH,

∴ D will coincide with H Lastly, ∵ A coincides with E, and D with H, ∴ AD will coincide with EH Hence ABCD coincides with EFGH, and is equal to it

10 The Δ formed by joining the middle points of the sides of an equilateral Δ, is also equilateral

Let ABC be an equilateral Δ, X, Y, Z are the middle points of the sides of Δ ABC ∴ AX=BX=BZ=CZ=CY=AY In Δs XBZ and YCZ, XB, BZ=YC, CZ, ∠XBZ=∠YCZ=∠ of an equilateral Δ, ∴ XZ=YZ (I 4) In Δs XAY and YCZ, XA, AY=YC, CZ, and ∠XAY=∠YCZ=∠ of an equilateral Δ, ∴ XY=YZ (I 4), XY=YZ=ZX, ∴ the Δ XYZ is equilateral

11 If in a Δ ABC, the ⊥ AD from the vertex A on the base BC bisects the base, then the Δ is isosceles

In Δs ABD, ACD, BD, DA=CD, DA and ∠BDA=∠CDA (being rt ∠s), ∴ AB=AC (I 4).

* *12 If a line be divided internally into unequal segments, the distance of the point of section from the middle point of the line, is half the difference of the segments, but if it be divided externally, the distance of the point of section from the middle point of the line, is half the sum of the segments*

Let AB be bisected in C, divided internally in D and externally in E Make AF=BD, and produce BA until AG=BE, ∵ AB is bisected in C and AF=BD, FC=CD, and FD is the diff of AD and DB Hence CD is half the diff of the internal segments AD and DB, ∵ AG=BE and AB is bisected in C, ∴ GC=CE, and GE is the sum of AE and EB, ∴ CE is half the sum of the external segments AE and EB

13 Given the sum and difference of two straight lines, find them

Let AB be the given sum, C the middle point of AB, and CD half the given difference of the required lines, then by (Ex 12 above) AD and DB are the required straight lines

14 The lines drawn from the angular points of an equilateral Δ to the middle points of the opposite sides, are equal

Let ABC be an equilateral Δ, and D, E, F be the middle points of AB, BC, CA Then in Δs BAF, CBD, BA=CB, and AF=BD, and ∠BAF=∠CBD=∠ of an equilateral Δ, ∴ BF=CD So it may be shown that BF=AE

15 If three points be taken on the sides of an equilateral Δ, namely, one on the each side, at equal distances from the Δ, the lines joining them form a new equilateral Δ

Let D, E, F be the points taken in the sides BC, CA, AB of the equilateral Δ ABC, so that BD=CE=AF It is required to

shew that the Δ DEF is equilateral In the Δs FBD, DCE, $\because$ FB=DC, and BD=CE, and $\angle$FBD=$\angle$DCE=$\angle$ of an equilateral Δ, $\therefore$ FD=ED So, it may be shown that FD=FE

16 The equal sides BA, CA of an isosceles Δ BAC are produced beyond the vertex A to the pts E and F, so that AE=AF, and FB, EC are joined, shew that FB=EC.

$\because$ BA=CA, AE=AF (Hyp), $\therefore$ BA + AE=CA + AF or EB=CF In Δs FCB and EBC, FC, CB=EB, BC and $\angle$ FCB =$\angle$EBC (I 5), $\therefore$ FB=EC

17 The mid pts of the sides of a square are joined Shew that the resulting quadrilateral has, all its sides equal

Let ABCD be a square E,F,G,H are the mid points of AB,BC, CD, DA, and EF, FG, GH, HE are joined. It is required to prove that EFGH has all its sides equal In Δs HAE and EBF, HA, AE=EB, BF, $\angle$HAE=$\angle$EBF (being rt $\angle$s), HE=EF (I 4) Similarly EF=FG and FG=GH, EFGH has all its sides equal.

18 From C any pt in a str line AB, CD is drawn at rt $\angle$s to AB, meeting a ◯ described with centre A and distance AB, in D, and from AD, AE is cut off=AC, shew that AEB is a rt $\angle$

In Δs BAE, DAC, $\because$ BA=DA, and AE=AC, and $\angle$BAE=$\angle$DAC, $\therefore$ $\angle$BEA=$\angle$DCA=a rt $\angle$

* *19 Upon the sides AB, BC and CD of a ▱m ABCD, three equilateral Δs are described, that on BC towards the same parts as the ▱m, and those on AB, CD towards the opposite parts Prove that the distance of the vertices of the Δs on AB, CD from that on BC, are respectively=the two diagonals of the ▱m* (Cam Ex. Pap 1862)

Let P, O, Q, be the vertices of the equilateral Δs Then $\angle$POQ=$\angle$PCD + $\angle$QCD,=$\angle$PCD+$\angle$PCB=$\angle$BCD, also PC =BC, and CQ=CD, $\therefore$ PQ=DB Again, $\angle$OBP=$\angle$OBA+$\angle$ABP=$\angle$PBC+$\angle$ABP=$\angle$ABC, and OB=AB and BP=BC, $\therefore$ OP=AC (I 4)

20 The str. lines which join the extremities of the base of an isosceles Δ to the mid pts of the opposite sides, are=one another

In the Δs DBC and ECB, DB, BC=EC, CB, and $\angle$DBC =$\angle$ECB (I 5), $\therefore$ CD=BE

21 If the $\angle$ ABC, ACB at the base of an isosceles Δ be bisected by the str lines BD, CD; shew that DBC will be an isosceles Δ.

Since the $\angle ABC = \angle ACB$ (I 5), $\therefore$ $\angle CBD = \angle BCD$, and hence $CD = BD$ (I 6) *i e*, Δ BCD is isosceles

22 In fig. of I 5, if FC and GB meet at O, prove that FO = GO

$\angle BFC = \angle CGB$, $BF = CG$, $BG = CF$, $\therefore$ $\angle BCF$ or $\angle BCO = \angle CBG$ or $\angle CBO$ (I 4), $\therefore$ $BO = CO$ (I 6) Also it is proved in (I 5) that $BC = CF$ and that $BO = CO$, $\therefore$ the remainder $FO =$ remainder GO

23. Two isosceles Δs XYZ, XYS stand on the same base, shew that $\angle ZXS = \angle ZYS$, (i) when the Δs stand on the *opposite* sides of the base.

$ZX = ZY$, $\therefore$ $\angle ZXY = \angle ZYX$. $SX = SY$, $\therefore$ $\angle SXY = \angle SYX$ (I. 5), $\therefore$ $\angle ZXY + \angle SXY = \angle ZYX + \angle SYZ$, *i e* $\angle ZXS = \angle ZYS$

(ii) When the Δs stand on the same side of the base.

Because $ZX = ZY$, $\therefore$ $\angle ZXY = \angle ZYX$ $\because$ $SX = SY$, $\angle SXY = \angle SYX$ (I 5), $\therefore$ $\angle ZXY - \angle SXY = \angle ZYX - \angle SYX$, *i e.* $\angle ZXS = \angle ZYS$

N. B Hence, in case second, if ZS be joined, ZS bisects the $\angle XZY$ For it has been proved in case second that $\angle ZXS = \angle ZYS$ In the two Δs ZXS, ZYS, $ZX, XS = ZY, SY$, and $\angle ZXS = \angle ZYS$, $\therefore$ $\angle XZS = \angle YZS$ (I 4)

24 The opposite $\angle$s of a rhombus, are bisected by the diagonals which join them

Let ABCD be a rhombus and AC its diagonal Then AC shall bisect $\angle$s BAD, BCD Then $\because$ in the Δs BAC and CAD, $AB = AD$, AC is common and $BC = CD$, hence $\angle BAC = \angle DAC$, and $\angle ACB = \angle ADC$ (I. 8), *i e* AC bisects $\angle$s BAD, BCD. Similarly it may be shewn that the diagonal BD bisects $\angle$s ABC, ADC

25 If two ⊙s cut each other, the line joining their pts of intersection, is bisected at rt $\angle$s, by the lines joining their centres

Let AB (line joining their centres) cut CD in E; then in Δs CAB, DAB, $\angle CAB = \angle DAB$ (I 8) and in Δs CAE, DAE, $CE = DE$ (I 4) and $\angle CEA = \angle DEA =$ a rt $\angle$.

26 If the opposite sides of a quadrilateral are equal, shew that the opposite $\angle$s are also equal.

Let ABCD be a quadrilateral, join AC and BD In the Δs ADC and ABC; $AD, DC = CB, BA$; AC is common, $\therefore$ $\angle ADC = \angle ABC$ (I 8)

27 ACB, ADB are two Δs on the same side of AB, such that

AC=BD, and AD=BC, and AD, BC intersect at O shew that the △ AOB is isosceles

In the △s BAD and ABC, AD=BC, AB is common, and BD=AC, hence ∠BAD=∠ABC (I 8) and ∴ OB=OA (I 6), i e △ AOB is isosceles

28 The diagonals of a rhombus or of a square, bisect each other ⊥ly

Let ABCD be a rhombus or a square, AC and BD the two diagonals ABD, CBD are two isosceles △s on the same base BD, AC bisects BD ⊥ly (I 8) Hence also BD bisects AC ⊥ly

29 The opposite ∠s of a rhombus are equal

Let ABCD be a rhombus Join BD In △s BAD, BCD, BA=BC, AD=CD, BD is common, ∠A=∠C Hence ∠ABC= ∠ADC

30 Two △s ABC, ABD on the same base AB, and on the opposite sides of it, are such that AC=AD, and BC=BD, shew that the line joining the points C and D cuts AB at rt ∠s in E

In △s CAB and DAB, CA=DA and AB is common, base BC=base BD, ∠CAE= ∠DAE (I 8) Again in △s CAE and DAE, CA=DA and AE is common, ∠CAE = ∠ DAE, ∴ ∠AEC= ∠AED (I 4), and since they are adjacent ∠s, each of them=a rt. ∠, ∴ CD is ⊥ to AB

31 In a quadrilateral ABCD, AB=AD, and BC =DC shew that the diagonal AC bisects each of the ∠s which it joins

△ ABC= △ ADC (I 8), ∴ ∠BAC=∠DAC, ∠BCA= ∠DCA

32 In a quadrilateral ABCD, the opposite sides AD, BC are equal, and also the diagonals AC, BD are equal if AC and BD intersect at K, shew that each of the △s AKB, DKC is isosceles

△ ABD= △ BAC (I 8), ∠ABD=∠BAC, ∴ △ AKB is isosceles (I 6) Similarly △ KDC is isosceles

33 Divide a given ∠ into 4 equal parts

Bisect the ∠, and then bisect the two halves of the ∠

34 Find a st line=half the sum *of two given st lines, and one=half their* difference

Let XY and XZ be two given st lines Place XY and XZ in the same st line, so that they are measured in opposite directions from X, and bisect YZ

Let XY and XZ be placed in the same straight line, so that they are measured in the same direction from X, and bisect YZ

35. Divide a given st line into four equal parts.

Bisect the st line and then bisect the halves of it

***36 From two given pts. to draw two equal st. lines, which shall meet in the same pt of a st. line given in position**

Let A and B the given points, and CD the given straight line Join AB, and bisect it in F, and from F draw FE at rt ∠s to AB meeting CD in E E is the point required Join AE, EB. Since AF=FB, and FE is common, and the ∠s at F are rt. ∠s, ∴ AE=EB

This can be otherwise enunciated thus —In a given st. line find a pt. that is equidistant from two given pts

N. B —When the two pts are so situated that the line joining them, is ⊥ to the given line, the problem is impossible, unless they happen to be equidistant from the line, for the line bisecting ⊥ ly the line joining them would be ‖ to the given line, and would therefore never meet it.

**37 From two given pts on the* same *side of a st line given in position, to draw two st lines which shall meet in that line, and make equal ∠s with it*

Let A and B be the given points, and DE the line given in position From A, let fall the ⊥ AD, and produce it to C, making DC=AD. Join CB, AP Then AP, PB will be the lines required ∵ AD=DC and DP is common, and the ∠s at D are rt ∠s, ∴ △ APD=△CPD, and ∠APD=∠CPD=the vertically opposite ∠ BPE

** 38 Through two given pts on* opposite *sides of a given st line, draw two st lines, which shall meet in that given st line, and include an ∠ bisected by that given st line*

Let AB be the given st. line, and P, Q the given pts Draw PD ⊥ AB, and produce it making DE=DP Join EQ, and produce it to cut AB at C; C shall be the point required Join CP, then, ∵ in the △s CDP, CDE, PD=DE, DC is common, ∠s at D are rt. ∠s; and ∴ equal, hence ∠ DCP = ∠ DCE (I 8).

N B —When the two points are so situated that the line joining them, is ⊥ to the given line, the problem is impossible, unless they happen to be equidistant from the line

**39 To find a point which is equidistant from the three vertical pts of a △ ABC*

Bisect the sides AB and BC at D and E (I 10), and through the points D and E, draw $\perp$s, and produce them until they meet at F The point F is at equal distances from A, B and C Draw FA, FB, FC, $\triangle$ BFA is isosceles, and also $\triangle$ BFC is isosceles Hence it is evident that FA=FC=FB

40 If two str lines cut one another, the four $\angle$s which they make at the point where they cut, are= four rt $\angle$s (*Cor 1. Prop I 15*)

For $\angle$AEC+ $\angle$AED=2rt $\angle$s (I 13), and $\angle$BED+$\angle$BEC= 2 rt $\angle$s (I 13), $\angle$AEC+$\angle$AED+$\angle$BED+$\angle$BEC=4 rt $\angle$s

41 All the successive $\angle$s made by any number of str lines meeting at one point, are together=four rt $\angle$s (*Cor 2 Prop I 15*)

Let OA, OB, OC, OD which meet at O, make the successive $\angle$s AOB, BOC, COD, DOA, it is required to prove these $\angle$s=4 rt $\angle$s, produce AO to E Then $\angle$AOB+$\angle$DOA= $\angle$BOC+ $\angle$ COD + $\angle$ DOA= ($\angle$AOB + $\angle$BOE) + EOD+ 2 rt $\angle$s+2 rt $\angle$s=4 rt $\angle$s

42 On a given base, describe an isosceles $\triangle$, such that the sum of its equal sides, may be equal to a given str line

Bisect the given st line, and with the half line for the equal sides, describe an isosceles $\triangle$ on the given base

43 If a str line drawn bisecting the vertical $\angle$ of a $\triangle$, also bisects the base, the $\triangle$ is isosceles

Let CD bisect the $\angle$ C, and base AB Produce CD, making DE=CD, join AE In the $\triangle$s ADE, BCD, AD = BD, CD=DE and $\angle$ADE= $\angle$BDC (I 15), BC=AE, $\angle$ DEA = $\angle$ DCB (I 4)= $\angle$ACD (hyp), $\therefore$ AC=AE (I 6)=BC, ABC is an isosceles $\triangle$

44 Construct an isosceles $\triangle$, having given the base and the length of the $\perp$ drawn from the vertex to the base

Bisect AB at D Through D, draw DC $\perp$ AB, making DC= X Join AC and BC In $\triangle$s ADC, BDC, AD, DC=BD, DC and $\angle$ ADC= $\angle$ BDC=(being rt $\angle$s), . AC=BC (I 4), $\therefore$ ABC is an isosceles $\triangle$ on AB, having CD=X

45. If any line be drawn through the middle pt of the line joining two given pts —any two pts in the former line, that are equidistant from the mid pt, are also equidistant from the two given pts

Let M, N be the given pts, O the mid pt of MN, and XY any line through O, also let OX=OY, then will MX=NY. For in the Δs OMX, ONY, the sides OX, OM, in the one, are respectively=OY, ON, in the other, and the vertical ∠s at O are equal (I 15), consequently the Δs are everyway equal (I 4) and ∴ MX=NY

46 ABC is a Δ, and the ∠ A is bisected by a st. line which meets BC at D. Shew that BA is greater than BD, and CA greater than CD

Since ∠ ADB > ∠ CAD (I 16), and ∴ it > ∠ BAD, ∵ ∠ CAD=∠ BAD, hence AB is > BD (I 18). In the same way, AC may be shewn to be > CD

47. ABCD is a quadrilateral, of which AD is its longest side, and BC the shortest. Shew that ∠ABC is > ∠ ADC and ∠ BCD > ∠ BAD

Join BD. Now ∠ ABD> ∠ ADB (I 18), ∵ AD>AB and ∠ CBD > ∠ BDC (I 18), ∵ CD > BC, hence ∠ ABC > ∠ ADC. Similarly ∠ BCD may be shewn to be > than ∠ BAD

48 Any three sides of a quadrilateral figure, are together greater than the fourth side

Let ABCD be a quadrilateral, join AC. Then AB+BC are > AC, ∴ AB+BC+CD are > AC+CD. But AC+CD are > AD, ∴ AB+BC+CD are > AD.

49. BC, the base of an isosceles Δ ABC, is produced to any point D, shew that AD is greater than either of the equal sides

∵ ∠ACB > ∠ ADC, ∠ABC > ∠ ADC, i e ∠ ABD > ∠ ADB and ∴ AD >AB or AC

50 If a st line be drawn through A one of the angular pts of a square cutting one of the opposite sides, and meeting the other side produced at F. Shew that AF is > than the diagonal of the square.

For ∠ at B is a rt ∠, ∠ACF is obtuse (I 16), and ∴ greater than ∠AFC; hence AF>AC

51 The hypotenuse is the greatest side of a right-angled Δ

From (I 17), it is evident, that any Δ must have at least, two acute ∠s, ∴ the rt ∠ is the greatest ∠, and has the greatest side opposite to it

52 In an obtuse-angled Δ, the greatest side is opposite the obtuse angle.

In the Δ ABC, let ∠ ABC, be obtuse. Then, each of the other

$\angle$s must be acute (I 17), and $\therefore$ AC must be greater than either of the other sides

53 *If the base of an* isosceles $\triangle$ *be produced both ways, the exterior* $\angle$ *s thus formed are equal*

The $\angle$s at the base are equal (I 5), and the exterior $\angle$s are the supplements of the $\angle$s at the base, $\therefore$ the exterior $\angle$s are equal

54 ABC is a $\triangle$, and the base BC is produced both ways If the exterior $\angle$ thus formed are equal, prove that $\triangle$ BAC isosceles

the exterior $\angle$s are equal, their supplements, namely the $\angle$s at the base, are also equal, the $\triangle$ is isosceles (I 6)

55 *From a pt outside a given st line, there can be drawn to the st line,* only one $\perp$

If possible, from C let there be drawn to the given straight line AB, two $\perp$s CD, CE Then CDE is a $\triangle$ and the exterior $\angle$CEB > interior opposite $\angle$ CDE, which is impossible, since they are both rt $\angle$s

56 If any side of a $\triangle$, is produced both ways, the exterior $\angle$s so formed, are together greater than two rt $\angle$s

Let ABC be a $\triangle$, the base BC is produced both ways, to E and D

$\angle$ABE + $\angle$ABC = 2 rt $\angle$s (I 13), also $\angle$ACD + $\angle$ACB = 2 rt $\angle$s (I 13), hence ($\angle$ABE + $\angle$ACD) + ($\angle$ABC + $\angle$ACB) = 4 rt $\angle$s But ($\angle$ABC + $\angle$ACB) < 2 rt $\angle$s, ($\angle$ABE + $\angle$ACD) > 2 rt $\angle$s

57 The perimeter of a quadrilateral, is greater than the sum of its diagonals

Let ABCD be a quadrilateral, AC, BD the two diagonals (AB + BC) > AC (I 20) also (AD + CD) > AC, and (BC + CD) > BD, and (AB + AD) > BD, 2 (AB + BC + CD + AD) > 2 (AC + BD), $\therefore$ (AB + BC + CD + AD) > (AC + BD)

58 The $\perp$ is the shortest line that can be drawn from a given pt to a given str line, and of others, that which is nearer to the $\perp$, is less than one more remote, and two, and only two equal str lines, can be drawn from the given pt to the given str line—one on each side of the $\perp$

Let A be the given pt and BC the given indefinite str line From A, let fall the $\perp$ AD, and draw any other lines AF, AG, AH &c of which AF is nearer to AD than AG, and AG than AH, AD shall be the least, AF < AG, and AG < AH

For in the Δ ADF, $\angle$ at D is a rt $\angle$ and consequently the $\angle$ at F is acute; hence AF $<$ AD, it may be shewn that AD is $<$any other str line drawn from A; hence AD is the shortest line

Again, $\because$ the $\angle$ at D is a rt. $\angle$, the $\angle$ AFG $>$ a rt. $\angle$ and $\therefore >$ $\angle$AGF, hence AG $>$ AF (I 19) In the same manner it may be shewn that AH $>$ AG

Lastly, from A there can only be drawn to BC, two str lines equal to each other. Make DE = DF, and join AE; then AE = AF; and besides AE no other line can be drawn = AF For, if possible, let AK = AE *i e* a line more remote is = to the nearer to the $\perp$, which is impossible, AK is not = AE, it may be shewn that no other line but AE can be = AF.

59. In a quadrilateral, if two opposite sides which are not || are produced to meet one another; shew that the perimeter of the greater of the two Δs so formed $>$the perimeter of the quadrilateral

Let ABCD be a quadrilateral, let AD and BC the non-parallel sides be produced to meet in E, forming a Δ ABE Now AB + BE + EA = AB + (BC + CE) + (ED + DA) = (AB + BC + DA) + (CE + ED); but (CE + ED) is $>$ DC, $\therefore$ (AB + BE + EA) is $>$(AB + BC + CD + DA)

60 The sum of the distances of any pt. from the three $\angle$s of a Δ, is greater than the semi-primeter of the Δ Prove the three cases, when the pt. is (1) *inside* the Δ, when it is (2) *outside*, and when it is (3) *on a side*

Let O be any point either *inside* or *outside* Δ ABC Then (AO + BO) $>$AB, (BO + CO)$>$BC, (CO + AO)$>$CA (I 20) $\therefore$ 2 (AO + BO + CO) $>$ (AB + BC + CA), (AO + BO + CO)$>$half of (AB + BC + CA)

When O is *on one of the side* as BC, the only modification to be made on the foregoing proof is to substitute CO + AO = CA, for (CO + AO)$>$CA.

61. The sum of the diagonals of a trapezium is$>$ than the sum of any four lines which can be drawn to the four $\angle$s, from any pt within the figure, except from the intersection of the diagonals

Let ABCD be a trapezium, whose diagonals are AC, BD cutting each other in E they are $<$ the sum of any four lines which can be drawn to the $\angle$s, from any other pt within the trapezium Take any point P, and join PA, PB, PD Then AC is$<$AP + PC, and BD$<$BP + PD, $\therefore$AC + BD are $<$AP + PB + PC + PD

62. The sum of the distances of any pt within the Δ from its angular pt is less than the perimeter of the Δ

Let ABC be the Δ, O the point within it Then (AO+BO) <(BC+CA), (BO+CO)<(CA+AB), (CO+AO)<(AB+BC) (I 21), 2(AO+BO+CO)<2(AB+BC+CA), (AO+BO+CO) < (AB+BC+CA)

63 The sum of the distances of any pt from the $\angle$r pts of a quadrilateral, is >half the perimeter of the quadrilateral

Let O be any pt within or outside, the quad ABCD Then (OA + OB)>AB, (OB+OC)>BC, (OC+OD)>CD, (OD+OA)>DA, 2 (OA + OB+OC+OD) > (AB + BC+CD+DA), (OA+OB+OC+OD)>½ (AB+BC+CD+DA)

64 In a Δ, any two sides are together > twice the str line drawn from the vertex to the mid pt of the base

Let ABC be the given Δ, and CM the st line joining the vertex to the mid pt of the base; AC + CB shall be >2 CM Produce CM to D making MD=CM, and join BD, then in the Δs AMC, BMD AM=BM, CM=DM and $\angle$AMC=$\angle$BMD (I 15), AC=BD (I 4) But in the Δ BCD, the sides DB+BC are >CD (I 20) and consequently AC+BC are CD or 2 CM

65 In any Δ, the sum of the *medians*, is less than the perimeter

Let ABC be a Δ, AH, BK, CL the three medians Then (AB + BC) > 2 BK, (BC+CA) >2 CL, (CA + AB) >2 AH, ∴ 2(AB+BC+CA) > 2 (AH +BK+CL), ∴ (AB+BC+CA) is> (AH+BK+CL)

66 Construct a quadrilateral, whose sides shall be equal to those of a given quadrilateral

Let ABCD be the given quadrilateral, join AC, make Δ EFG, having EF=AB, FG=BC, GE=CA, and on EG on the other side from F, make Δ EGH, having GH=CD, and HE=AD Then EFGH is the required quadrilateral

** 67 Construct a rectilineal figure whose sides shall be equal to those of a given rectilineal figure*

Let ABCDE be the given rectilineal figure Join AD, make a quad FGHK, having FG=AB, GH=BC, HK=CD, KF=DA, on FK on the other side from G and H, make Δ KLF, having KL =DE, LF=EA Then FGHKL is the required rectilineal figure

68 Find a pt in a given str line, the difference of the

distances of which from two given pts on the same side of the line shall be the greatest possible (Cam. Ex. Pap 1865)

Let AB be the given st line, P and Q the given points

Join PQ and produce it to meet AB, or AB produced in R Then shall PQ be>the difference of any two lines drawn from P and Q to the same point in AB, as PD, QD For PQ+QD is>PD, PQ > the difference between PD and QD

69 Two st lines are drawn to the base of a Δ from the vertex, one bisecting the vertical ∠, and the other bisecting the base Prove that the latter is the greater of the two st lines

Let ABC be a Δ, having side AC > AB Let AD, the bisector of ∠ BAC, meet BC in D, and let AE bisect BC in E From AC, cut off AF=AB and join FD Then ∵ AB=AF and AD is common, and ∠BAD=∠FAD, ∠ADF=∠ADB, and ∴ ∠ADE>∠ADB, ∠ADE>∠AED, ∵ AE>AD

70 On a given base AB, describe a Δ whose remaining sides shall be=two given str. lines X, and Y. Point out how the construction fails, if, any one of the three given lines, is greater than the sum of the other two.

With A as centre, and a radius=X, describe a ⊙ CFM, with B as centre and a radius=Y, describe a ⊙ CEM intersecting the ⊙ at C and M Then ABC (or ABM) shall be the Δ required

N B The construction fails, when any one of the three given straight lines, is greater than the sum of the other two (I 20) Cf Notes on I 22, Pt I

71 If one ∠ of a Δ is=the sum of the other two, the Δ can be divided into two isosceles Δs

Let ∠ C of the Δ ABC, be=the sum of the other two ∠s At the pt C make ∠ BCD = ∠ CBD (I 23) and ∴ BD = CD (I. 6), *i e*, the Δ BCD is isosceles Hence ∠ ACB = ∠ ABC+∠CAB, and ∠ BCD= ∠ CBD, hence ∠ ACD=∠CAD and ∴ AD = CD (I 6), *i e*, the Δ ACD is isosceles

72 If the ∠C of a Δ is=the sum of the ∠s A and B, the side AB is=twice the st line joining C to the mid pt of AB

This follows *directly* from the preceding exercise

73 ABC is a Δ, in which the vertical ∠ BAC is bisected by the st line AX. from B draw BD ⊥ to AX, and produce it to meet AC or AC produced in E, shew that BD=DE

In Δs ADE, ADB, ∠ EAD = ∠ BAD, ∠ ADE = a rt. ∠ = ∠ ADB, and AD is common, ∴ DE = DB (I 26)

74 Any point on the bisector of an ∠, is equidistant from the arms of the angle

Let X be any pt on the bisector of the ∠ BAC, XM and XN ⊥s on AB and AC Then in the Δs AXM, AXN, ∠XAM = ∠ XAN, ∠ M = a rt angle = ∠ N, side XA is common XM = XN (I 26)

75 If BX and CY, the bisectors of the ∠s at the base BC of an isosceles Δ ABC, meet the opposite sides in X and Y shew that the Δ YBC = Δ XCB

In Δs YBC, XCB, ∠ B = ∠ C, ∠ BCY = ∠ CBX, side BC in common, Δ YBC is identically = Δ XCB (I 26)

76 In a given st line, to find a point equally distant from two given st lines In what case is this impossible ?

When the lines form a Δ ABC, draw AD bisecting the ∠ A opposite to the side BC, in which the point is to be found Draw DE, DF ⊥s to AC, AB respectively Then ∵ ∠ A is bisected, ∠ E = ∠F = a rt ∠, and AD is common to Δs AED, AFD, ∴ they are = in every respect (I 26), DE = DF

If all the positions of these lines be considered, it will be seen that the problem fails—when the lines are ||l and at unequal distances

77 Through a given pt draw a str line such that the ⊥s on it from two given pts —may be on opposite sides of it and equal to each other

Let B, C be the points, from which ⊥s are to be drawn to a line through A (another given point) Join BC, through its middle point D, draw ADN, and drop the ⊥s BM, CN on it Then BN, CN shall be equal For, in the Δs BMD, CND, the ∠s at D are equal (I 15), the ∠s at M, N are rt ∠s, and BD = DC, hence BM = CN (I 26)

78 Construct a Δ, having given the base, one of the ∠s at the base, and the sum of the other two sides

Take BC = the given base, at B make ∠ CBD = the given ∠, cut off BD = the sum of the other sides, join CD, and at C, make ∠ DCA = ∠ BDC Let CA meet BD at A Then ABC is the required Δ Since ∠ ACD = ∠ ADC, ∴ AC = AD (I 6), ∴ BA + AC = BD = the given sum of sides

79 Given two ∠s of a Δ, and the side adjacent to them, construct the Δ

Let AB be the given side. At A make ∠ CAB = one of the

given ∠s. At B make ∠ DBA = the other given ∠; AC and BD being drawn, so as to lie on the same side of AB. Then AC, BD will, if produced, meet in some point E. Hence ABE will be the required △.

80. In a △ ABC, the vertex A is joined to X, the mid. pt of the base; shew that ∠ AXB is obtuse or acute, according as AB is greater than or less than AC

First, let BA > AC: in two △s AXB and AXC; sides AX, XB = AX, XC; base AB > base AC: ∴ ∠ AXB > ∠ AXC (I. 25). But ∠ AXB + ∠ AXC = 2 rt. ∠s (I. 13), of which *one* is greater than the *other*; ∴ ∠ AXB is *obtuse*.

Secondly, Let AC > AB: ∴ ∠ AXC > ∠ AXB (I. 25), hence ∠ AXB is *acute*.

81. Construct a △—having given the base, an ∠ at the base, and the difference of the other two sides.

Take BC = the given base: at B make ∠ CBD = the given ∠ (I. 23): cut off BD = the difference of the other sides: join CD, and at C make ∠ DCA = the supplement of ∠ BDC. Let CA meet BD produced at A. Then ABC is the required △. Since ∠ ACD = ∠ ADC; ∴ AC = AD (I. 6): ∴ BA—AC = BD = given difference of sides.

82. The diagonals of a rhombus are unequal.

Let XAYZ be a rhombus: let ∠ AXZ > ∠ XAY. In △s XAZ, XAY: sides XA, XZ = XA, AY: ∠ AXZ > ∠ XAY. ∴ AZ > XY. (I. 24).

83. M is the middle point of the base BC of an isosceles △ ABC, and N is a pt. in AC. Shew that the difference between MB and MN is less than that between AB and AN.

MB—MN = MC—MN, and ∴ less than CN (Ex. 7, p. 38 Text) *i.e.*, less than AC—AN *i.e.*, less than AB—AN.

84. If a st. line meet two or more ∥ str. lines, and is ⊥ to one of them: it is also ⊥ to all the others.

Let AB, CD be the ∥ str. lines. Draw any line EFG ⊥ DC meeting AB in F. Then EFG is also ⊥ AB. Since AB is ∥ DC, ∴ ∠ EFB = ∠ FED = a rt. ∠ (I. 29), hence EF is ⊥ AB.

85. If two str. lines A and B are respectively ∥ to two others C and D; shew that the inclination of A to B is = that of C to D.

Produce B and C to meet, if necessary, at G; then ∠ AEB = ∠ EGF = ∠ GFD (I. 29).

86 Any st line drawn ∥ to the base of an isosceles △, makes equal ∠s with the sides

Let ABC be an isosceles △, and DE ∥ to its base BC The ∠s ADE, AED, shall be equal For ∠ ADE = ∠ ABC (I 29) = ∠ ACB (I 5) = ∠ AED

A similar proof will apply, when the ∥ cuts the sides, when they are produced through the vertex

87 From a given pt. C, draw a str line CD that shall make with a given str line AB, an ∠ = a given ∠M

Through C draw CE ∥ to AB (I 31) At the pt C in EC, make ∠ ECD = ∠ M Let CD meet AB in D. Now ∠ ECD = ∠CDA, ∠ CDA = ∠ M

88 If the str line which bisects an exterior ∠ of a △ is ∥ to the base, show that the △ is isosceles

Let the str line AE, bisecting the exterior ∠ DAC of the △ ABC, be ∥ to its base BC Then △ABC shall be isosceles For ∠ CBA = ∠DAE (I 29) = ∠ EAC (Hyp) = ∠ACB (I 29), hence AC = AB (I 6), *i e* △ ABC is isosceles

89 If from any pt in the bisector of an ∠, a str line is drawn ∥ to either arm of an ∠—the △ thus formed, is isosceles

O is any point on BD, the bisector of ∠ ABC, and OX is drawn ∥ AB Now ∠ *OBX* = ∠ ABO = ∠ *BOX* (I 29), BX = OX

90 From X, a point in the base BC of an isosceles △ ABC, a str line is drawn at rt ∠s to the base cutting AB in Y, and CA produced in Z, show that △AYZ is isosceles.

∠ YBX + ∠ BXY + ∠ XYB = 2 rt ∠s (I 32) = ∠ CZX + ∠ ZXC + XCZ (I 32), but ∠ YBX = ∠ ACB or ∠ XCZ (for △ ABC is isosceles) and ∠ BXY = ∠ ZXC = a rt ∠, ∴ ∠ XYB = ∠ CZX, but ∠ XYB = ∠ AYZ, ∠Z = ∠Y, AZ = AY, ∴ △ AYZ is isosceles

91 If the alternate extremities of two equal and ∥ st lines be joined, the connecting lines, bisect each other

Let MN, KL, be two equal and ∥ lines, then ML and KN, joining their alternate extremities, bisect each other in O In △s MON, KOL, the vertical ∠s at O are equal (I 15), the alternate ∠s at M and L are equal, and KL = MN, ∴ the △s are equal in every respect (I 26), and consequently, OK = ON, and OL = OM *i e* ML and KN are bisected in O

92 *Through a given point, to draw a st line that shall be equally inclined to two given st lines*

Let AB, AC, be two given st lines, and P a given point, it is reqd to draw from P a st line PC, making equal ∠s with AB and AC Produce CA to F, and bisect ∠ BAF by AG, and from P draw PC ||| AG Then PC is the reqd line Since AG and PC are ||| ∴ the exterior ∠ FAG = the interior and opposite ∠ACP (I 29), ∠GAB = ∠ABC, as they are alternate ∠s, but ∠FAG = ∠GAB (constr), ∴ ∠ ACP = ∠ABC

When the point lies between the given st lines, or is situated in one of them, it is evident that the same method of solution applies

93. If two sides of a △ are unequal, and if from their point of intersection, three st lines are drawn namely, the bisector of the vertical ∠, the median, and the ⊥ to the base, the first is intermediate in position and magnitude to the other two

Let ABC be a △, and from A, let there be drawn AD ⊥ BC, AE bisecting ∠BAC, and AF bisecting BC it is required to prove that AE lies between AD and AF If AB = AC, the three straight lines AD, AE, AF coincide

Since, the sides are unequal, let AB > AC, so that ∠ B < ∠ C (I 18) Then ∠ BAD the complement of ∠ B > ∠ CAD the complement of ∠ C (I 32), ∵ AE the bisector of ∠A will lie between AD and AB From AB cut off AH = AC, join HE. Then in △s HAE, EAC; AH = AC (const.), AE common, and ∠HAE = ∠EAC (Hyp), ∴ HE = EC, ∠AHE = ∠ACE (I. 4). But ∠ ACE or ∠ACD is the complement of ∠ CAD, ∴ ∠ACE is *acute*, ∴ its equal ∠ AHE is also *acute*, hence ∠ BHE is *obtuse* (I 13), ∵ BE > HE or EC, AF (the median), lies between AE (the bisector of ∠ A) and AB (the greater side)

Again AE (the hypotenuse of △ ADE) > AD, and in △ AEF, ∠AEF is *obtuse* {∵ being exterior ∠ of △ADE, ∠AEF > ∠ ADE (I. 16), and ∠ ADE = a rt ∠ and ∠AFE *acute*, ∴ AF > AE (I 19)} Thus AE is intermediate in magnitude and position to AD and AF

94. A st line drawn between two |||s and terminated by them, is bisected Shew that any other st line passing through the mid pt and terminated by the |||s is also bisected at that point

Let BD be the st line terminated by the |||s AB, CD Through its mid pt E, let AEC be drawn terminated by the |||s, AC shall be bisected at E.

In the Δs AEB and CED, $\angle$BAE = $\angle$ECD (I 29), $\angle$AEB = $\angle$CED (I 15), BE = ED (hyp), hence AE = CE (I 26)

95 If through a pt equidistant from two ||l lines, two st lines are drawn cutting the ||ls, the portions of the latter thus intercepted—are equal

Let the st lines AB, CD be ||ls, and E a pt equidistant from them Through E draw the lines LER, PEQ Then LP shall be = QR Draw MEN at rt $\angle$s to the ||ls In the Δs LME, RNE, $\angle$ MLE = $\angle$ ERN (I 29), $\angle$ LEM = $\angle$ REN (I 15), and ME = EN (hyp), hence LM = RN (I 26) For similar reasons, PM = QN Hence (LM + MP) = (RN + NQ) or LP = QR

96 Construct an isosceles Δ of given altitude, whose sides shall pass through two given pts and have its base on a given st line

Let FG be the given st line and D, E, be the pts Draw NL ||l to FG and at a distance = the given altitude of the Δ (I 31) In NL, find a pt C such that DC, EC make equal $\angle$s with NL Produce CD, CE to meet FG in A, B respectively Then ABC is the required Δ For its vertex C being in NL, its altitude is of given magnitude, also the sides CA, CB pass through the given pts D, E, and its base is on the given st line FG And, $\angle$NCA = $\angle$CAB, $\angle$LCB = $\angle$ CBA (I 29), and equal to each other (const), $\angle$ CAB = $\angle$ CBA, hence Δ ABC is isosceles (I 6)

97 Trisect a rt angle

Let FAB be rt $\angle$. Take any distance AB, and on AB describe an equilateral Δ ABC (I. 1), bisect $\angle$ CAB by AH (I 9) Then $\angle$CAB is = one-third of two rt $\angle$s or = two-thirds of one rt $\angle$ (I 32), $\angle$FAC = $\angle$ CAH = $\angle$ HAB = one-third of a right angle

98 Trisect a given finite straight line

Let AB be the given st line On it describe an equil Δ ABC (I 1), bisect $\angle$s CAB, CBA by AD, BD meeting in D (I 9) and draw DE, DF ||l to CA and CB respectively Then AB shall be trisected in E and F $\because$ ED is ||l to AC, $\angle$ EDA = $\angle$DAC = $\angle$DAE, $\therefore$ AE = ED For similar reasons, DF = FB But DE being ||l to CA and DF ||l to CB, the $\angle$ DEF = $\angle$CAB, $\angle$DFE = $\angle$CBA, and $\therefore$ $\angle$EDF = $\angle$ACB and hence the Δ EDF is equiangular and hence equilateral, $\therefore$ DE = EF = FD, and hence AE = EF = FB, $\therefore$ AB is trisected

**99 ABCD is a square and E a point in BC, str line EF is drawn at rt $\angle$s to AE, and meets the str line, which bisects the*

∠ between CD and BC produced in a pt F. Prove that AE = EF. (Cam. Ex. Pap 1870)

Take in AB a part BX = BE, and AX = EC. Then ∠ FEC + ∠AEB = a rt. ∠ = ∠AEB + ∠BAE. ∴ ∠FEC = ∠XAE, also ∠AXE = ∠XBE + ∠XEB = $\frac{3}{2}$ of rt. ∠ = ∠ECF. Since ∠FEC = ∠ XAE, and ∠ECF = ∠AXE and AX = EC; ∴ FE = AE (I.26).

101 *Through two given pts. draw two st lines, forming with a st line, given in position, an equilateral* △. (Cam. Ex. Pap. 1853.)

Let A, B be the pts and BM the line Draw AC ⊥ CB; make ∠CAN = $\frac{2}{3}$ of a rt. ∠. Then ∠ CNA = $\frac{1}{3}$ of a rt ∠. Draw DE ∥ BC, make ∠ EDF = $\frac{2}{3}$ of a rt ∠, ∴ ∠ OFN = $\frac{2}{3}$ of a rt ∠. ∴ △ FON is an equilateral △

102 *If ABC be a △, in which C is a rt ∠, show how to draw a st. line ∥ to a given st line, so as to be terminated by CA and CB and bisected by AB* (Cam. Ex Pap 1861.)

Let EF be the given straight line Draw EP ⊥ CB, making an acute ∠ with EF Make ∠NCB = ∠EFP, CN meeting AB in N. Draw QNM ∥ EF Then ∠NCM = ∠EFP = ∠NMC; ∴ CM = NM Also ∠ QCN = complement of ∠ NCM = complement of ∠ NMC = ∠ CQN. ∴ QN = NC = NM

*103 *To bisect a ▱m by a st. line drawn from a point in one of its sides*

Let ABCD be a ▱m, and P a given pt. in AB Draw the diameter BD, which bisects the ▱m Bisect BD in F, join PF, and produce it to meet DC in E. PE bisects the ▱m Since ∠ PBD = ∠ BDE, and the vertically opposite ∠s at F are equal and BF = FD, ∴ △ PBF = △ EFD But △ ABD = △ BDC, ∴ APFD = BFEC, and to these equals, adding the equal △s DFE, PFB, the figure APED = PECB; and AC is ∴ bisected by PE.

103 If from any pt in the diagonal (or diagonal produced), of a ▱m, str lines be drawn to the opposite ∠s, they will cut off equal △s.

From any point E in AC, the diagonal of the ▱m ABCD, let lines EB, ED be drawn Then △ABE shall be = △AED, also △ BEC, shall be = △ CED Draw the diagonal BD intersecting AC in F. The bases BF, FD being equal, the △s BFA, DFA as also the △s BFE, DFE are equal (I 38). hence △s BAE, DAE are equal And △ ABC being = △ ADC, the △s BEC, DEC are also equal (Cf Text Book P. 110, Ex. 11.)

+104. Given the base, difference of sides, and difference of the base ∠s; construct the △ *(Cf. Text, p 108, Ex 11)*

Let BC be the given base, draw CE making the ∠ BCE = ½ the given difference of base ∠s, and from the centre B, at distance = to the given difference of sides, describe a ( ) cutting CE in D and E, join BD, and produce it to A. Draw CA making ∠ DCA = ∠CDA, and meeting BD produced in A. Then ABC is the required Δ. For AD = AC (I 6), ∴ ∠ADC = ∠ACD. Now ∠ ACB − ∠ ABC = (∠ ACD + ∠ DCB) − ∠ ABC = (∠ADC + ∠DCB) − ∠ ABC = ∠ABC + ∠DCB + ∠DCB − ∠ABC (I 32) = 2∠DCB or 2 ∠BCE, ∴ ∠DCB is half the difference of the base ∠s ACB, ABC, and BD is the difference of the sides, ∴ Δ ABC has the given base, given difference of sides, and given difference of base angles.

105. Given the base, sum of the sides, and difference of the base ∠s construct the Δ.

Let BC be the given base. Draw CE, making the ∠ BCE = half the given difference of base ∠s, and draw CF ⊥ to CE. From B as centre, and radius = the given sum of the sides, describe a ◯ cutting CF in F. Bisect DF in A, and join AC. Then ABC is the required Δ, for DA = AC = AF.

106. The two Δs formed by drawing st. lines from any point within a ▭m, to the extremities of two opposite sides, are together = half of the ▭m. (*Cf. Text Bk. Page 74, Ex. 4*)

Let P be any point within the ▭m ABCD, from which let lines PA, PD, PB, PC be drawn to the extremities of the opposite sides. The Δs PAD, PBC are = ½ the ▭m, as also the Δs APB, DPC. Through P draw EPF ||| AD or BC, then Δ APD is = ½ of AEFD, and BPC is = ½ of BEFC, (I 41), ∴ Δs APD, BPC are together = ½ of ABCD. So, if a line be drawn through P, ||| AB or DC, it may be shewn that Δs APB, DPC are together = ½ of ABCD.

***107. In a rt. ∠ d Δ, the middle point of the hypotenuse, is equidistant from the three angles, and conversely.**

In rt. ∠d Δ ACB let O be the middle pt. of the hypotenuse AB. Draw ON ||| to BC, to meet AC in N. Join CO.

Since O is the middle pt. of AB and ON is ||| to BC, ∴ N is the middle pt. of AC, (*Ex. 1 p. 96, Text*). And ∠s at N being rt. ∠s, ∴ Δ ONC = Δ ONA, hence, OC = OA = OB.

Converse. If in a Δ ABC, pt. O in AB is such that OA = OB = OC. Then ∠ ACB shall be a rt. ∠.

Now ∠ OAC = ∠OCA and ∠ OBC = ∠ OCB (for OA = OB = OC), ∴ ∠ACB = ∠ CAB + ∠CBA, ∴ ∠ ACB = a rt ∠ (I 42 Cor).

108 Each of the base ∠s of an isosceles △ = ½ the exterior vertical ∠.

The exterior vertical ∠ is = the sum of the base ∠s. But the base ∠s are equal, ∴ each of them, is half of the exterior vertical ∠

***109. In a rt ∠d △, if a ⊥ be drawn from the rt. ∠ to the hyp, the △s on each side of it, are equiangular to the whole △, and to one another (Cf VI 8)**

Let ABC be a rt ∠d △, having ∠ BAC a rt ∠ and from A let AD be drawn ⊥ to BC. Then must △s DBA, DAC be equiangular to △ ABC, and to each other

Since rt ∠ BDA = rt ∠ BAC, and ∠ ABD = ∠ CBA, ∴ ∠ DAB = ∠ ACB (I 32), ∴ △ DBA is equiangular to △ ABC So, it may be shewn that △ DAC is equiangular to △ ABC, ∴ they are equiangular to one another

110 If two st. lines are ⊥s to two other st lines, each to each, the acute ∠ between the first pair is = the acute ∠ between the second pair

Let AB, AD be respectively ⊥s to CB, CD

In the two △s AEB, CED, ∠AEB = ∠CED (I 15), ∠ABE = a rt ∠ = ∠CDE, ∠ BCD = ∠BAD (I. 32)

**111 The area of a △ is = ½ that of the rectangle under an altitude and its corresponding base*

Let AN be an altitude of △ ABC Draw BP, CQ ||| AN, and PAQ ||| BC Then △ ABC = ½ rect. PBCQ = ½ rect under AN, BC. And similarly for either of the other altitudes.

112 Construct a rt ∠d △, having given the hypotenuse and the difference of the sides

Take any str line terminated at A, and make AB = diff of the sides. Produce AB, and at B make ∠ CBP = ½ a rt ∠, and with centre A, and radius = the hypotenuse, describe a ⊙ cutting BP at P From P let fall the ⊥ PM on AC, and join AP; then APM shall be the △ required

For in the △BMP, ∠BMP is a rt ∠, and ∠ MBP = half a rt ∠, hence ∠BPM is = ½ a rt ∠ (I 32), and ∴ PM = BM (I 6) Now since PM = BM, ∴ the difference of AM, MP is = AB, the given difference of the sides

113. Construct a rt. ∠d △, having given the perimeter, and the acute ∠.

Let AB be the given perimeter of the Δ to be constructed At A make ∠ABC=½ the given acute ∠, and at B make ∠ABC=½ its complement At C (where AC, BC meet), draw CD making ∠ACD=∠CAD, and CE making ∠BCE=∠CBE (I 23) Then CDE shall be the required Δ

For ∠ CDE=∠DAC + ∠ ACD (I 32), and equal to the given ∠ Similarly it can be shewn that ∠CED is its complement, ∠ DCE is a rt ∠ (I 32) Again, since CD= AD (I 6) ∴ CD+ DE + EC=AB, the given perimeter

114 If the sides of an equilateral and equiangular pentagon *be produced to meet, the ∠s formed by these st lines are together= 2 rt ∠s*

Let ABCDE be an equilateral and equiangular *pentagon*, and let the sides be produced to meet in F, G, H, I, K, the angles at these points are together=2 rt ∠s

For since ∠BCG is exterior ∠ of the Δ FCI, it is=∠s at F and I For the same reason the ∠ CBG = ∠s at K and H; and ∴ the ∠s at F, G, H, I, K, are=the three ∠s of the ΔBCG, *i e* =two rt ∠s

* *115 If the sides of an equilateral and equiangular* hexagon *be produced to meet, the ∠s formed by these st lines, are together=* four *rt ∠s*

Let ABCDEF be an equilateral and equiangular *hexagon*, and let the sides be produced to meet in G, H, I, K, L, M, the ∠s at these points are together=4 rt ∠s

For GLI being a Δ, the ∠s at G,I,L, are together=2 rt ∠s, and for the same reason, the ∠s at H,K,M, are together =2 rt ∠s; ∴ the six ∠s are=4 rt ∠s

116 The difference of the ∠s at the base of any Δ, is double the ∠ contained by a line drawn from the vertex, ⊥r to the base, and another bisecting the ∠ at the vertex

From B the vertex of the Δ ABC, let BE be drawn ⊥ to the base, and BD bisecting ∠ABC, the difference of the ∠s BAC,BCA is=2 ∠EDB The ∠ BAC=diff of the ∠s BEC and ABE, *i e* of a rt ∠ and ∠ ABE (I 32) Also ∠ BCA=the diff of a rt. ∠ and ∠ EBC, ∴ the diff of the ∠s BAC and BCA=the diff of the ∠s ABE and EBC, *i e* (since ∠ ABD= ∠ DBC) = 2 ∠ EBD.

117. If from one of the equal ∠s of an isos Δ, any line be drawn to the opposite side, and from the same pt a line be drawn to the opposite side produced,

so that the part intercepted between them may be = to the former, the ∠ contained by the side of the Δ and the first drawn line, is double of the ∠ contained by the base and the latter.

Let ABC be an isos Δ, having the side AB=AC From B draw any st line BD, and also BE cutting off DE=DB, ∠ ABD is=2 ∠ CBE For ∠ DCB is=the two ∠s DEB, CBE, *i e* is=the two ∠s DBE, CBE or= ∠DBC and 2 ∠ CBE, but ∠ DCB = ∠ ABC, ∴ ∠ ABC = ∠ DBC and 2 ∠ CBE, and taking away ∠ DBC, (which is common to both), ∠ ABD = 2 ∠ CBE

118 Through two given pts, draw two str lines forming with a str line, given in position, an equilateral Δ

Let A, B be the given pts and CD a st line given in position In CD, take any st line DE, and on it, describe an equilateral Δ DEF (I 1) Through A, B draw GAC, GBH ||| to EF, DF (I 31) Then CGH shall be the Δ reqd For ∠ GCH=∠ FED, and ∠ GHC = ∠ EDF (I 29), ∴ ∠CGH=∠EFD (I. 32) The Δ CGH being equiangular, is equilateral

119 From a given pt, it is required to draw two ||| st lines, two equal st lines, at rt. ∠s to each other

Let A, be given pt *without*, and FG, BH the given ||| st lines From A draw AC ⊥ to the ||ls In BJ take BC=AD, and in DG take DE=AC Join AB, AE Then AB, AE shall be the lines required For in Δs ACB and ADE, BC=AD, AC=DE, and ∠ ACB= ∠ ADE=a rt ∠, hence AB = AE, and ∠ CBA = ∠ DAE (I 4) Now since ∠ DAF = ∠ CBA and ∴ adding ∠ BAC to these equals, ∠ BAE is=the ∠s BAC, ABC But the ∠s BAC, ABC are=one rt ∠ (I 32), ∴ ∠ BAE is also a rt ∠

120 AB and CD are two st lines intersecting at D, and the adjacent ∠s so formed, are bisected, if through any pt X in DC, a st line YXZ be drawn ||| to AB and meeting the bisectors in Y and Z, shew that XY=XZ.

Now ∠ XZD=alt. ∠ZDB (I 29)=∠ ZDX, ∴ ZX=DX. So ∠ XYD=∠ YDA (I 29)=∠ YDX, ∴ YX=XD, ∴ XY=XZ

121. AB is the hypotenuse of a rt ∠ d Δ ABC, find a point D in AB, such that DB may be=the ⊥ from D on AC

Bisect ∠ ABC by BE meeting AC at E, and through E, draw ED ||| to BC Then D shall be the pt. reqd For ∠AED=∠BCE

(I. 29)=a rt ∠, hence DE is at rt ∠s to AC Also since, ∠DEB = ∠EBC (I 29)= ∠EBD (const), hence BD=DE (I 6).

122 ABC is an isos. Δ, find pts. D and E in the equal sides AB, AC, such that BD=DE=EC

From (Ex 121),DE=BD Again, since AB=AC, and AD=AE (I 6), for ∠ADE= ∠AED (Ex 86), hence BD=EC But DB=DE (Ex 121), ∴ BD=DE=EC

123 If one ∠ of a ▭m, is a rt ∠, all its ∠s are rt ∠s

Let ABCD be a ▭m Since the adjacent ∠s A and B are together=two rt ∠s (I 29), if one of them is a rt ∠ (Hyp), the other must also be a rt ∠, and the ∠s C and D are rt. ∠s, being=their opposite ∠s A and B (I 34)

124 If the opposite sides of a quadrilateral are equal, the figure is a ▭m *(See notes on Prop I 34)*

125 If the opposite ∠s of a quadrilateral are equal, the figure is a ▭m *(See notes on Prop I 34)*

126 If the diagonals of a quadrilateral bisect each other, the figure is a ▭m *(See notes on Prop I 34)*

127 The ▭DABE has all its sides equal and one ∠ =a rt ∠, prove that all its ∠s are rt ∠s

The opposite sides AB and DE are ||l, and they are met by the line AD, ∴ ∠s A and D are together=two rt ∠s (I 29), but ∠A is a rt ∠, ∴ ∠D is a rt ∠ And ∵ in the ▭m DABE, the ∠s E and B are respectively opposite to ∠A and ∠D, which are rt ∠s. ∴ ∠E and ∠B, are rt ∠s, and ∴ all the ∠s of the ▭ DABE are rt ∠s

128 The diagonals of a ▭m, bisect each other

Let ABCD be a ▭m, and AC, BD its diagonals, AC, BD shall bisect each other For, in the Δs AEB and DEC, ∠ABE= ∠EDC (I 29), and ∠AEB= ∠DEC (I 15), and AB=CD, hence AE=EC and BE=ED (I 26)

129 If the str line joining two opposite ∠s of a ▭m, bisect the ∠s, the figure is a rhombus. *Or*, If two opposite ∠s of a ▭m, are bisected by the diagonal which joins them, the figure is equilateral

Let ABCD be a ▭m, and AC the line joining its opposite ∠s If AC bisect the ∠s at A and C, then ABCD shall be a rhombus. ∵ ∠BAD= ∠BCD and AC bisects them, ∴ ∠BAC= ∠BCA, ∴ AB=BC (I 6) But AB=CD and BC=AD, ∴ AB=BC=CD= DA Hence ABCD is a rhombus

*130 **Consruct an isos. rt ∠d △, having given (1) the sum of the hypotenuse and side, (2) their difference**

(1) Take CD = the sum of the hypotenuse and one side From D in DC, draw DB making ∠ CDB = ½ of a rt. ∠, and meeting CB (the ⊥ from C to DC) in B Draw BA making ∠ DBA = ∠ ADB Then ABC is the △ reqd For ∠ CAB = ∠ ADB + ∠ ABD (I. 32) = 2 ∠ ADB, = ½ a right angle (const); ∴ ∠ CBA = ½ a rt ∠ (I. 32), the ∠ C being rt ∠. Hence CA = CB (I. 6), and DA = AB, also CA + AB = DC

(2) Take CE = the given diff Draw EB making the ∠ CEB = ¼ of a rt ∠, and meeting the ⊥ CB in B; draw BA making ∠ EBA = ∠ AEB and meeting EC produced in A. then ABC is the △ reqd. For ∠ AEB = ∠ ABE (const), ∴ AB = AE. Hence CE is the difference of the hypotenuse AB and a side AC, also ∠ s AEB, ABE are each = ¼ of a rt ∠, ∴ ∠ BAC = ½ a rt. ∠ (I. 32), ∴ ∠ ABC = ½ a rt. ∠, and AC = BC (I. 5)

131. ABCD is a rhombus, and the diagonal AC is bisected at O If O is joined to the angular pts B and D: shew that OB and OD are in one st line

∵ ABCD is a rhombus ∴ BA = BC, ∠ BAC = ∠ BCA (I 5), AO = CO In △ s BAO and CBO, sides BA, AO = BC, CO and ∠ BAO = ∠ BCO ∴ ∠ BOA = ∠ BOC = a rt ∠. Similarly in the △ s ADO and CDO, ∠ AOD = ∠ COD (I 4) = a rt. ∠, ∴ ∠ AOB + ∠ AOD = 2 rt ∠ s, ∴ OB and OD are in *one* straight line (I 14)

132. In fig I 9, shew that the bisector of ∠ BAC, bisects the ∠ DFE

AD = AE (const) DF = FD being sides of an equilateral △, AF is common; ∴ AF, FD = AF FE, and base AD = base AE ∴ ∠ AFD = ∠ AFE (I. 8), ∴ ∠ DFE is bisected by AF.

133. The st line which bisects the vertical ∠ of an isos △, also bisects the base

Let BAC be a △, AD bisects the ∠ BAC, so that ∠ BAD = ∠ CAD In △s BAD, CAD; BA = CA, AD is common, ∠ BAD = ∠ CAD; ∴ BD = CD (I 4); thus AD bisects BC.

134. Pts. D and E in the base BC of an isos. △ AOC, are equidistant from its extremities, i. e BD = CE Shew that AD = AE

In △s ABD, ACE, DB, BA = EC, CA, and ∠ DBA = ∠ ECA, ∴ AD = EA (I. 4)

135 XYZ is an isos. △, having XY = XZ, and the

∠s at Y and Z are bisected by str lines which meet at P prove that PX bisects the ∠ YXZ

∠ *PYZ* = ½ XYZ = ½ XZY = ∠ *PZY*, ∴ PY = PZ (I. 6), ∴ ∠XYP = ∠ XZP In Δs XYP and XZP, XY = XZ, XP is common and PY = PZ, ∴ ∠ YXP = ∠ ZXP (I 8), *i e* XP bisects ∠ YXP

136 In Δ ABC, if AC is not > AB, shew that any st line AM drawn through the vertex A, and terminated by the base BC, is < AB

Suppose AC is not > AB, hence AC is either < AB or AC = AB It is to be proved that, in both cases AM < AB

(*Case I*) Here AC<AB, ∴ ∠ ACB> ∠ ABC But ∠ AMB > ∠ ACM (I 16), ∴ ∠ AMB is much more > ∠ ABM or ∠ ABC, ∴ AB>AM

(*Case II*) If AC = AB, ∴ ∠ ACB = ∠ ABC (I 5), ∠AMB > ∠ ACM and ∴ ∠ AMB > ∠ ABM ∴ AB > AM

**137 Given a pt and three st lines, two of which are ||l, to find a pt in each of the ||ls, that shall be equidistant from the given pt, and such that the st line joining them, shall be ||l to the other given st line*

Let AB, CD, AE, be the 3 st lines, of which the first two are ||l, and let P be the given pt, it is reqd to find points B and C such that BC may be ||l AE, and that PB may = PC From P draw PE ⊥ AE, meeting CD in D and AB at F, bisect DF in O, draw COB ||l AE, then B and C are the reqd pts Join PB and PC, then in 2 Δs BOF, COD OD = OF (const), the ∠s at O are equal (I 15), and ∠ BFO = ∠ CDO (I 29), ∴ the Δs are every way equal, and ∴ CO = OB, again ∠ COD = ∠ AEF (I 29), and the ∠s at E are rt ∠s (const), hence ∠ COD and ∠ BOD are rt ∠s and equal (I 13), also CO = OB, and OP is common to the 2 Δs POB, POC, ∴ the Δs are every way equal (I 4), hence PB = PC The points B, C, are ∴ equidistant from P, and the line BC joining them, is ||l AE

***138 The line joining the mid pts of two sides of a Δ, is ||l to the base and = to the *half* of it**

(See p 96, Ex 2 Text)

*** 139 The quad[1] formed by joining the successive mid pts of the sides of a given quad[1], is a □m**

Let ABCD be any quad[1] and EH, HG, GF, FE, lines joining the successive mid pts of its sides, then shall FGHE be a □m. For let AC, DB be the diagonals of the given fig Now EH is

‖l AC (Ex 138) the base of the Δ ACD So FG is ‖l AC (Ex 38) Hence FG and EH are ‖ls (I 30), and for a similar reason EF and GH are ‖l (I 30) Hence EFGH is ▭m

140 ABC is a Δ, in which OB, OC bisect the $\angle$s ABC, ACB, respectively, shew that, if AB is $>$ AC, then OB is $>$ OC

Since AB>AC, ∴ $\angle$ACB $>$ $\angle$ABC (I 18), $\angle$OCB is> $\angle$OBC (being their halves), ∴ OB is >OC

141 If the st lines bisecting the $\angle$s at the base of an isos Δ, be produced to meet, they will contain an $\angle$ =an exterior $\angle$ of the Δ

Let BE, CE, bisect the $\angle$s at the base of the isos Δ ABC, meet at E. The $\angle$ BEC shall be= the exterior $\angle$ ACD. For $\angle$ECD = $\angle$CBE + $\angle$BEC (I 32), and $\angle$ECD = $\angle$ACE + $\angle$ACD, and $\angle$ ACE = $\angle$CBE being halves of the $\angle$s at the base of an isos Δ, hence $\angle$ACD = $\angle$BEC

142 If the base of any Δ is produced both ways, shew that the sum of the two exterior $\angle$s diminished by the vertical $\angle$, =two rt $\angle$s

Let ABD be a Δ, base BD is produced both ways to C and E To prove $\angle$ABC + $\angle$ADE − $\angle$BAD = 2 rt $\angle$s

Now $\angle$ABC = $\angle$BAD + $\angle$ADB (I 32), $\angle$ADE = $\angle$BAD + $\angle$ABD (I 32), $\angle$ABC + $\angle$ADE = 2$\angle$BAD + $\angle$ADB + $\angle$ABD, $\angle$ ABC + $\angle$ADE − $\angle$BAD = $\angle$ BAD − $\angle$ADB + $\angle$ABD = 2rt $\angle$s

143 The sum of the sides of an isos Δ, is $<$ the sum of the sides of any other Δ on the same base and between the same ‖ls

Let ABC be an isos Δ, and ADB any other Δ on the same base and between the same ‖ls AB, ED, AC + BC will be< AD + DB Since EC is ‖l AB, the $\angle$ECA = $\angle$CAB, and also $\angle$DCB = $\angle$CBA, but $\angle$ CAB = $\angle$ CBA, ∴ $\angle$ECA = $\angle$ DCB AC and BC drawn from two given pts A and B, on the same side of ECD given in position, make equal $\angle$s with the line (I 6), ∴ AC + CB<any other two lines AD + BD, drawn from the same points to that line

144 Draw a st line at rt $\angle$s to a given finite st. line, from one of its extremities, without producing the given st line

Let AB be the given str line On AB describe any isosceles Δ ABC Produce BC to D, making CD=BC. Join AD Then shall AD be ⊥ to AB Now $\angle$ACD + $\angle$ACB = 2 rt $\angle$s

(I 13) = 2 ∠ CAB + 2 ∠ DAC (I 32) = 2 (∠ CAB + ∠ DAC) = 2 ∠ DAB, ∠ DAB = a rt ∠

145 The ∠s of a quadrilateral are = 4rt. ∠s

Let ABCD be a quadl, its ∠s at A, B, C, and D are = 4 rt ∠s For draw the diagonal DB, then ∠s A, ABD and ADB, of Δ DAB, are = 2 rt ∠s (I 32), and ∠s C, CDB, and DBC of Δ DCB, are = 2 rt ∠s (I 32), but ∠ ABC is composed of the two ∠s ABD, DBC, and ∠ ADC is composed of the two ∠s ADB, CDB, hence ∠s A, C, ABC, and ADC, of the quadrilateral, are = the ∠s of the two Δs = 4 rt ∠s (I 32)

146 If two ▭ms have one ∠ of the one = one ∠ of the other, the ▭ms are equiangular to one another

Let ABCD, EFGH be two ▭ms, having ∠ A = ∠ E, since ∠ A + ∠ B = 2 rt ∠s (I 29) and ∠ E + ∠ F = 2 rt ∠s (I 29) ∴ ∠ A + ∠ B = ∠ E + ∠ F, ∠ B = ∠ F Now ∠ A = ∠ C, and ∠ E = ∠ G (I 34), ∠ C = ∠ G And ∠ B = ∠ D, and ∠ F = ∠ H (I 34) ∠ D = ∠ H

147 Of two ▭ms, which are between the same ||ls, that is the greater, which stands on the greater base

Let ABCD, EFGH be two ▭ms between the same ||ls, AH, BG, but let the base BC be > FG From BC cut off BK = FG, and through K draw KL || AB meeting AB at L, then ▭m ABKL = ▭m EFGH (I 36) But ▭m ABCD > ▭m ABKL, ▭m ABCD > ▭m EFGH

148 If the diagonals of a ▭m are equal, all its ∠s are rt ∠s

Let ABCD be a ▭m, diagonal AC = diagonal BD (hyp) Then all its ∠s shall be rt ∠s Here Δ ABD = Δ CBA (I 8) ∠ DAB = ∠ CBA = a rt ∠ (I 29) Similarly the ∠ D and ∠ C, can be proved to be rt ∠s

149 Two rectangles are equal, if two adjacent sides of one, are = two adjacent sides of the other, each to each *(Apply I 4 or I 34)*

150 In a ▭m, which is not rectangular, the diagonals are unequal

Let ABCD be a ▭m, which is not rectangular Join AC, BD In Δs DAB, CBA, DA, AB = CB, BA Then DB >, =, < AC, according as ∠ DAB is >, =, < ∠ ABC (Cf Ex 82)

151 In a ▭m, the ⊥s drawn from one pair of opposite ∠s, to the diagonal which join the other pair, are equal

In $\triangle$s ABM, DCN; $\angle$ AMB = a rt. $\angle$ = $\angle$ CND; $\angle$ ABM = $\angle$ CDN (I. 29); AB=CD, $\therefore$ AM=CN (I. 26)

152. Any st. line drawn through the mid. pt. of a diagonal of a □m, and terminated by a pair of opposite sides, is bisected at that pt.

Let ABCD be a □m; and BD one of diagonals. MN is drawn through the mid. pt. of BD. Then MN shall be bisected at O. In $\triangle$s MOB and NOD, $\angle$ MOB = $\angle$ NOD (I. 15), and since MB is ∥ ND, $\therefore$ $\angle$ MBO = $\angle$ NDO, and BO=DO, $\therefore$ MO=ON (I. 26).

153. The $\angle$ contained by the bisectors of the $\angle$s at the base of any $\triangle$ is = the vertical $\angle$ + ½ the sum of the base $\angle$s.

Let BE and CE bisect the base $\angle$s of any $\triangle$ ABC meeting one another at E. Join AE and produce it to cut the base BC at D: $\angle$ CED = $\angle$ EAC + $\angle$ (ECA or ½ $\angle$ C) (I. 32), $\angle$ BED = ($\angle$ EAB + $\angle$ EBA or ½ $\angle$ B), $\therefore$ $\angle$ BEC = $\angle$A + ½ $\angle$s (B+C).

154. MNXY is a □m, and A, B, respectively the mid. pts of the sides MN and YX. Show that the figure MAXB is a □m.

In the $\triangle$s MYB and XNA, MY, YB=XN, NA, $\angle$ MYB = $\angle$ XNA, $\therefore$ $\angle$ MBY = $\angle$ XAN. But $\angle$ XAN = $\angle$ AXB (I. 29), $\therefore$ $\angle$ AXB = $\angle$ MBY (Ax. 1), $\therefore$ MB ∥ AX (I. 28). And AM ∥ XB (Hyp.); $\therefore$ MAXB is a □m.

155. From one of the $\angle$s of a □m, to draw a st. line to the opposite side, which shall be = that side together with the segment of it, which is intercepted between the st. line and the opposite $\angle$.

Let ABCD be the □m, A the $\angle$ from which the st. line is to be drawn. Produce DC to E, making CE=CD. Join AE, and at A make $\angle$ EAF = $\angle$AEF; then AF is the line reqd. For CE being=CD, EF=DC+CF, and $\angle$ FEA, $\angle$FAE, FA=FE, and $\therefore$ AF = DC + CF.

Cor.—In the same manner, if CE=CB, AF=EF=BC+CF.

156. On the base of a given $\triangle$, construct a second $\triangle$, = in area to the first, and having its vertex in a given st. line.

Let ABC be the given $\triangle$ on the base AB, and EF the given str. line. Through C draw CD ∥ AB (I. 31), meeting EF at D. Join BD, AD. Then ABD shall be the $\triangle$ reqd. Since CD is ∥ AB, $\therefore$ $\triangle$ ABC = $\triangle$ ABD (I. 37).

N B—If the base be of *given length* and Δ be given, we can construct a 2nd Δ = the given Δ by (Ex 21, p 111, Text)

157 If in the sides of a square, at equal distances from the 4 ∠s, 4 other points be taken, one in each side, the figure, contained by the str lines which join them, shall also be a square

Let E, F, G, H be four points at equal distances from the ∠s of the square ABCD Join EF, FG, GH, HE, then EFGH shall be a square. Since AH=EB, and AE=BF, and the ∠s at A and B are rt ∠s, HE=EF and ∠AEH = ∠BFE So it may be shown that HG=HE, and GF=EF, the fig HEFG is equilateral. It is also rectangular, for since the exterior ∠FEB=interior ∠s EBF, EFB, parts of which ∠AEH and ∠EFB are equal, the remaining ∠ FEH=remaining ∠ FBE and ∴ is a rt ∠ So it may be shown that ∠s at E, G, are rt ∠s, and EFGH, being equilateral and rectangular, is a square

158 Describe an isos Δ = in area to a given Δ, and standing on the same base

Let DBC be the Δ on base BC Bisect BC at E Draw EA at rt ∠s to BC Through D, draw DA ||| BC meeting EA at A. Join AB, AC, then ABC is the isosceles Δ required

159. A ▭m is divided by its diagonals, into 4 Δs of equal area.

Let ABCD be a ▭m, and let AC and BD its diagonals intersecting each other at O. Now AO=OC and BO=OD (Notes on I 34), ΔAOB = Δ COB = Δ COD = Δ AOD (I 38)

160 The three st lines which join the mid pts of the sides of Δ, divide it into 4 Δs, which are identically equal in area

Let M, N, P be the mid pts of the sides AB, BC, CA of any Δ ABC Join MN, NP, PN. For, since MP is ||| to BC, MN is ||| to AC, NP is ||| to BA (Ex 138), ∴ MBNP, MNCP, MNPA are ▭ms, Δ MBN = Δ MNP = Δ PNC, for similar reasons Δ AMP = Δ MNP = Δ PNC Thus ΔAMP = Δ MNP = ΔMBN = Δ PNC

N B—Δ MNP = ¼ of Δ ABC

161 Construct a rt ∠d isos Δ = to a given square

Let ABCD be the given square Join AC Draw AE ⊥ AC, meeting CD produced in E Then ACE is the Δ required Now ΔADC = Δ ADE (I 26), ∠ACE = ½ ∠ BCD = ½ a rt ∠, ∠ CAE = a rt ∠; ∠ AEC = ½ a rt ∠, ∴ ∠ AEC = ∠ ACE, ∴ AC=AE and ABCD = 2 Δ BCD = ΔACE

162. Construct ***a rhombus*** **= a given ▱m, and standing on the same base. When does the construction fail?**

Let MNXY be a ▱m standing on NX. From centre N, with NX as radius, describe a ⊙ intersecting MY at A. Join NA and draw XB ∥ NA, meeting MY produced at B. Then ANXB is the rhombus required, and it is = MNXY (I. 35).

N. B.—The construction fails, when the base, on which the rhombus is to be constructed, is < the altitude of the ▱m.

163. Describe a square = the sum of two given squares.

Let AB and X be the length of the sides of the two given squares. Draw BC ⊥ AB, making BC = X. Join AC. Then AC^2 shall be the square required. ∵ $AB^2 + X^2 = AB^2 + BC^2 = AC^2$ (I. 47).

164. Describe a square, which shall be = the difference of two squares, whose sides are given.

Take a line AB terminated at A, and cut off AO = a side of the greater, and OB a side of the lesser square. With O as centre, and radius OA, describe a ⊙ OCD, and from B, draw BC ⊥ to AD. Then BC^2 is the square required. Join OC, BC^2 = the difference of the squares on OC and OB, *i. e.* on AO & OB.

165. The square described on the diagonal of a given square is double of the given square.

Let ABCD be a square. BD its diagonal. $BD^2 = BC^2 + CD^2$ (I. 47). But BC = CD, ∴ $BD^2 = 2\ BC^2$ or $= 2\ CD^2$.

166. The ∠ contained by the bisectors of two adjacent ∠s of a quadrilateral, is = half the sum of the remaining ∠s.

Let ABCD be a quadl., AO and DO the bisectors of two adjacent ∠s A and D, meet at O. To prove ∠ AOD shall be = ½ ∠s (B+C). Now ∠s (BAD+ADC+C+B) = 4 rt. ∠s (Ex. 145), ∠s (ODA+OAD + DOA) = 2 rt. ∠s, ∴ ∠s (ODA + OAD + DOA) = ½ ∠s (ADC+BAD+B+C), or ∠ ODA + ∠ OAD + ∠ DOA = ½ ∠ADC + ½ ∠BAD + ½ ∠ s (B+C), ∴ ∠ DOA = ½ ∠s (B+C).

167. The ∠ contained by the bisectors of 2 exterior ∠s of any Δ, is = ½ the sum of the 2 corresponding interior ∠s.

Let ABC be a Δ, AB and AC are produced to D and E. Let CO and BO, the bisectors of two exterior ∠s ECB and DBC of Δ ABC. To prove ∠ O = ½ ∠s (ABC+ACB), ∠s (A+B+C) = ∠s (O+OBC+OCB) = 2rt. ∠s. (*a*) (I. 32). But ∠OBC = ½ ∠DBC

$=\frac{1}{2}$ $\angle$s (A+ACB) (I. 32), and $\angle$OCB$=\frac{1}{2}\angle$ECB$=\frac{1}{2}\angle$s(A+ABC) (*b*) $\therefore$ From (*a*) and (*b*), we have $\angle$A+ $\angle$s (B+C)$=\angle$A+$\frac{1}{2}$ $\angle$s (ACB+ABC) + $\angle$O, $\frac{1}{2}$ $\angle$s (ABC+ACB)= $\angle$O

168 To describe a $\square$m, the area and perimeter of which shall be respectively = the area and perimeter of a given Δ

Let ABC be the given Δ Produce AB to D, making BD=BC, bisect AD in E (I 10), draw BF $\parallel$ AC (I 31), and with centre A and radius AE, describe a $\frown$ cutting BF in G Join AG, and bisect AC in H (I 10) Join BH Draw HF $\parallel$ AG (I 31) Then AGFH shall be the $\square$m reqd For HF=AG=AE, $\therefore$ HF+AG=AD =AB+BC, and since GF=AH=HC, $\therefore$ GF+AH=AC, the perimeter of AGFH=the perimeter of Δ ABC, and $\square$m AGFH =2 Δ ABH (I 41) and $\therefore$ is= Δ ABC (I 38)

169 Describe a $\square$m equal a given square standing on the same base, and having an $\angle$=half a rt $\angle$

Let XYNM be the given square Join YM $\angle$ MYN=$\frac{1}{2}$ a rt $\angle$ Through N draw NP $\parallel$ YM (I 31), meeting XM produced at P Then $\square$MYNP=sq XYNM (I 35) having $\angle$MYN=$\frac{1}{2}$ a rt $\angle$.

**170 To describe a square which shall be = the sum of any number of given squares*

Let AB be a side of one of the given squares From B draw BC $\perp$ AB, and = a side of the 2nd square. Join AC, and from C draw CD $\perp$ to it, and = a side of the 3rd square Join AD and from D, draw DE $\perp$ AD and = side of the 4th Join AE Then AE^2 shall be$=AB^2+BC^2+CD^2+DE^2$ Since $\angle$s ADE, ACD, ABC are rt $\angle$s $\therefore$ $AE^2=AD^2+DE^2=AC^2+CD^2+DE^2=AB^2+BC^2+CD^2+DE^2$ And by proceeding in the same manner, whatever be the number of given squares, one = their sum, may be found

**171 Inscribe a square in a given rt $\angle$d isos Δ*

Let ABC be a rt $\angle$d isos Δ, having BA = BC Trisect the hypotenuse AC in the points D, E, and from D, E draw DF, EG $\perp$ to AC, join FG, then DFGE be the square required Since $\angle$ DAF $=\frac{1}{2}$ a rt $\angle$, and $\angle$ D = a rt. $\angle$, $\therefore$ $\angle$ DFA is $=\frac{1}{2}$ a rt. $\angle$ = $\angle$ DAF, hence DF = DA So it may be shown that, EG = EC But AD = EC, and $\therefore$ FD=DE = EG, and FG = DE (I 34), $\therefore$ the figure is equilateral, and it is rectangular (I 46), since the $\angle$s at D and E are rt $\angle$s, $\therefore$ it is a square

172 Construct a rhombus = a given $\square$m

Let ABCD be a $\square$m, from $\angle$ B, draw BE to meet the opposite side DC produced in E, and make BE=AB, and draw AF from

the adjacent $\angle$A ||| to BE. meeting DC, in F Then ABEF shall be the rhombus required Since AF = BE = AB (const.)=EF, the ▭m ABEF is a rhombus Also ABEF = ABCD (I. 35).

173. If from the extremity of the base of an isosceles Δ, a st line=one of the sides be drawn, to meet the opposite side, the $\angle$ formed by this st line and the base produced, is=three times either of the equal $\angle$ s of the Δ

Let ABC be an isos Δ, having AB=AC From C to AB (produced if necessary) draw CD = AC, and let BC be produced, $\angle$DCE = 3 $\angle$ ABC Since CA = CD, $\angle$ CAD = $\angle$ CDA, ∴ $\angle$CDA + 2 $\angle$ ABC = 2 rt. $\angle$s = $\angle$ CDA + $\angle$ CDB, hence $\angle$CDB = 2 $\angle$ ABC Now $\angle$ DCE=$\angle$CDB + $\angle$ CBD (I. 32), and consequently is = 3 $\angle$ ABC.

174 To bisect a given Δ, by a st line drawn from one of its $\angle$ s

Let ABC be the given Δ and A the $\angle$, from which the bisecting st line is to be drawn Bisect the opposite side BC in D, and join AD, AD shall bisect the Δ For since the base BD = base DC; ∴ Δ ABD = Δ ADC (I 38)

175 To bisect a given Δ, by a st. line drawn from a given pt in one of its sides (See p 113, Ex 36, Text.)

Let ABC be the given Δ, and P the given pt Bisect BC in D, join AD, PD, and from A draw AE ||| to PD, join PE, then PE shall bisect the Δ ABC Since AE ||| PD Δ APD = Δ EPD (I 37) from each of them take away Δ PFD, ∴ Δ AFP = Δ EFD Also BD = DC, Δ ABD= Δ ADC, parts of which Δs EFD, AFP are equal, ∴ ABEF=PFDC, hence ABEF + Δ AFP or ABEP = PFDC + Δ FED or Δ PEC, and ∴ Δ ABC is bisected by PE

176 To trisect a given Δ, from a given pt within *it.*

Let ABC be the given Δ, and P the given point within it. Trisect BC in D and E, join PD, PE, and from A draw AF, AG respectively ||| to them Join PF, PG, AP, then they shall divide the Δ into 3 equal parts Join AD, AE. Since AF is ||| PD, Δ APF = Δ ADF, to each of these, add Δ ABF, ∴ APFB = ΔADB. So APGC=ΔAEC, and the remaining ΔFPG= Δ DAE Now the Δs ABD, ADF, AEC, being on equal bases and of the same altitude, are equal (I 38 *Cor* I), ∴ APFB=ΔPFG =APGC and ∴ the Δ ABC is *trisected*

177. If two exterior $\angle$s of a Δ be bisected, and from the point of intersection of the bisecting st lines, a st line be drawn to the opposite $\angle$ of the Δ; it will bisect that $\angle$.

Let the exterior ∠s EBC, BCF, of the Δ ABC be bisected by BD, CD meeting in D Join DA, then it shall bisect ∠ BAC Let fall the ⊥s DE, DF, DG Then ∠ DBE = ∠ DBG & ∠ E = ∠ G = a rt ∠, and DB common to the Δ DBE, DBG, DE=DG So DG=DF, and DE = DF Hence in the rt ∠d Δs DAE, DAF, DE = DF, and DA is common, ∴ the Δs are equiangular, and ∠ DAE = ∠ DAF *i e*, ∠ BAC is bisected by AD

178 To bisect a trapezium by a st line drawn from one of its ∠ s *(Cf Text, p 113, Ex 39)*

Let ABCD be the given trapezium, and A the ∠ from which it is to be bisected Draw the diagonal AC, BD, and bisect BD, which is opposite to the ∠ A, in E Join AE, CE, and through E draw FEG ∥ AC (I 31) Join AG, then AG shall bisect the trapezium Since DE=EB, Δ AED = Δ AEB Also Δ DEC=Δ BEC, ∴ the fig AECD=the fig AECB Also Δ AEG= Δ CEG (I 38) take away the common part EHG, and ∴ Δ AEH= Δ GHC To the fig AECD, add Δ GHC, and take away its equal Δ AEH, and to AECB, add Δ AEH, and take away Δ GHC Now Δ AGB=the trapezium AGCD, or the given trapezium ABCD is bisected by AG

179 To bisect a trapezium by a st line drawn from a given pt in one of its sides

Let ABCD be the given trapezium, and P the given point Join PA, and from ∠ P, bisect the trapezium APCD by the st. line PE (Ex 178) On PE make Δ PEF= Δ ABP (See Ex 156) Bisect EF in G, join PG Then PG shall bisect the trapezium Since FG=GE, Δ PGF= Δ PGE But PGE=½ Δ ABP, and PEC=½ the fig PABC, hence PGC=½ of the trapezium ABCD which is bisected by PC

180 If two sides of a trapezium be ∥, then the Δ contained by either of the other sides, and the two st lines drawn from its extremities to the bisection of the opposite side, is ½ the trapezium *(Cf Ex 7, p 109, Text)*

Let ABCD be a trapezium, having the side AB ∥ to DC Let AD be bisected in E join BE, CE, then Δ BEC shall be ½ of the trapezium Through E, draw FEG ∥ to BC, meeting CD in G, and BA produced in F The alternate ∠ FAE=alternate ∠EDG the ∠s at E, being equal, and AE=ED, ΔACF=ΔDEG, hence the ▭m BFGC = trapezium ABCD But BFGC and Δ BEC, being on the same base BC, and between the same ∥s BC, FG, ∴ Δ BEC=½ of BFGC (I 41), and ∴ = ½ of ABCD.

N B—From the proof, it appears that a trapezium which has two sides ∥, may be reduced to a *▭m*=to it, by drawing through

the point of bisection of one of the sides, which are not ||l to the other of those sides, and meeting the ||l sides

181 Trisect a Δ, by st lines drawn from a given pt. in one of its sides

Let ABC be the given Δ, and X the given point in the side BC Trisect BC at the points P, Q (Ex. 98)

Join AX, and through P and Q draw PH and QK ||l to AX. Join XH, HK Then XH and XK shall trisect the Δ Join AP, AQ Now Δ ABP = Δ APQ (I 38) = Δ ACQ = $\frac{1}{3}$ Δ ABC. Again Δ BHX = Δ ABP (P 111, Ex 21, Text) = $\frac{1}{3}$ Δ ABC Also Δ CKX = Δ ACQ (P. 111 p Ex 21, Text) = $\frac{1}{3}$ Δ ABC, the remainder AKXH = $\frac{1}{3}$ Δ ABC

182 Of all ▭ms, which can be formed with diagonals of given length, the rhombus is the greatest (Cam. Ex Pap 1856)

Let ABCD be a rhombus, whose diameters AC, BD intersect in O Then ∠s at O are rt. ∠s The area of rhombus = rect AO BD Let ABCD be any ▭, not a rhombus, having its diagonals = that of the rhombus Then, if AP be drawn ⊥ BD, area of ▭ ABCD = rect. AP BD, and AP is < AO, ∴ area of the ▭ is < area of the rhombus.

183. To divide a given st. line into any number of equal parts

Let AB be the given st line. From A draw an indefinite st line AG, in which take any point E, and take EF, FG, etc each = AE, until the numbers of equal parts in AG = the number of parts into which AB is to be divided Join GB, and draw EC, FD ||l GB then AB shall be divided by these ||ls, in C and D, as reqd Draw EH Since FK is ||l AB, and ∴ ||l to one another Also since AC and EH are ||l and AF meets them, ∴ ∠EAC = ∠ FEH and ∵ EC, FD are ||l and AG meets them, ∴ ∠AEC = ∠ EFH, also AE = EF, ∴ AC = EH and EC = FH (I 26); but ED is a ▭m (constr), ∴ EH = CD and ∴ AC = CD So, from the Δs EFH and FGK, it is proved that CD = DB Hence AB has been divided into the reqd number of equal parts

184 The area of a rhombus, is = half the rect contained by the diagonals (Cal Ex Pap. 1862)

Let ABCD be a rhombus The diagonals AC and BD bisect each other, and cut at rt ∠s in E (I 41) Now ∵ Δ ABD = $\frac{1}{2}$ BD AE, and Δ BDC = $\frac{1}{2}$ BD CE, ∴ Δ ABD + Δ BDC = $\frac{1}{2}$ BD (AE+CE) or rhombus ABCD = $\frac{1}{2}$ BD. AC

185 If a st line DME be drawn through the middle point

M of the base BC of a △ ABC, so as to cut off equal parts AD, AE from the sides AB, AC, produced if necessary, respectively, then shall BD be = CE (Cam Ex Pap 1860)

Draw CP ||l to BD Then ∵ ∠ MBD = ∠ MCP, (I 29) and ∠ DMB = ∠ PMC (I 15) and BM = MC, ∴ CP = BD Now ∠ AED = ∠ ADE, and ∠ CPE = ∠ ADE (I 29), ∴ ∠CEP = ∠ CPE, and ∴ CE = CP = BD

186 The sum of the ⊥s drawn from any point within an equilateral △ to the 3 sides is = the ⊥ drawn from any one of the angular points to the opposite side (*Cal Ex Pap 1858*)

Let ABC be the equilateral △, and P the given pt within it From P draw PE, PF and PG ⊥rs to AB, BC, and AC respectively, and from the vertex A draw AD ⊥ to BC Join PA, PB, PC Then AD shall be = PE + PF + PG, ∵ △ BPC = ½ BC PF, △ APC = ½ AC PG, △ APB = ½ AB PE, ∴ △ PBC + △ APC + △ ABP or △ ABC = ½ BC (PF + PG + PE), since AB = BC = AC (being sides of an equilateral △) But △ ABC = ½ BC AD, ∴ ½ BC AD = ½ BC (PF + PG + PE), AD = PE + PG + PF

**187 ABC is an isosceles △, of which A is the vertex, AB, AC are bisected in D and E respectively, BE, CD intersect in F, shew that the △ ADE is = three times △ DEF* (Cam Ex. Pap 1857)

△EFC + △EFD = △EDC = ½ △ADC = ¼ △ ABC Also △ DFB + △EFD = △DEB = ½ △ ABE = ¼ △ ABC, △ EFC + △DFB + 2 △ EFD = ½ △ ABC, △ AFE + △ AFD + 2 △ EFD = ½ △ABC, △ ADE + 3 △ EFD = ½ △ ABC Also △ ADE = ¼ △ ABC ∴ 3 △ EFD = ¼ △ ABC, △ ADE = 3 △EFD

188 If st lines be drawn from the ∠s of any ▭m, ⊥ to any st line which is *outside* the ▭m, the sum of those from one pair of opposite ∠s, is = the sum of those from the other pair of opposite ∠s

Let the ⊥s DM, AN, CP, BQ be drawn from the angular points of the ▭m ABCD to any st line MQ which is *outside* the ▭m Then DM + BQ shall be = CP + AN Through A, B draw KR, BL, ||l MQ and ∴ ⊥ to DM, CP Let KR meet CB produced at R, then in △s AKD, and BLC, ∠AKD = ∠BLC, ∠ KAD = ∠LBC, each being = ∠ ARB (I 29), and AD = BC, ∴ KD = CL (I 29) To each of these equals, add KM, BQ or their equals AN, PL, then DM + BQ = CP + AN

**189 ABC, ABD are two equal △s upon the same base AB*

and on opposite side of it, join CD, meeting AB in E. Show that CE=ED. (Cam. Ex. Pap. 1851).

Let ABC, ABD be two equal △s upon the same base AB, and on opposite sides of it; join CD meeting AB in E. Make ∠ABX=∠ABD, and BX=BD, and join CX, AX, EX. Then △ABX=△ABD=△ABC. Hence CX is ∥ to BA (I. 39, 40); ∴ △CEB=△XEB on same base=△DEB (constr.) Hence △s CEB, DEB, must be on equal bases, CE, DE. For if not, let CE=EF. Then △CEB=△BEF (I. 38), which is impossible; ∴ CE=DE.

190 (a) In the figure I 1, if the ⊙s intersect at F, and if CA, CB are produced to meet the ⊙s in P and Q respectively; show that *(a)* P, F, Q are in the same st. line; and *(b)* shew that the △ CPQ is equilateral.

(a) △ ABF is equilateral (I. 1); ∠CAF=∠CAB+∠BAF $=\frac{1}{3}$ of 2 rt. ∠s $+\frac{1}{3}$ of 2 rt. ∠s$=\frac{2}{3}$ of 2 rt. ∠s: ∴ ∠PAF $=\frac{1}{3}$ of 2 rt. ∠s. Also ∠APF=∠AFP, for AP=AF, being radius of the same ⊙. Again ∠PAF+∠APF+∠AFP=2 rt. ∠s (I 32)$=\frac{1}{3}$ of 2 rt. ∠s +2 ∠APF=2 rt. ∠s; ∴ 2 ∠APF $=\frac{2}{3}$ of 2 rt. ∠s; ∴ ∠APF, and ∠AFP each$=\frac{1}{3}$ of 2 rt ∠s. For similar reasons, ∠BFQ$=\frac{1}{3}$ of 2 rt. ∠s, and since ∠AFB $=\frac{1}{3}$ of 2 rt. ∠s ∴ ∠AFP+∠AFB+∠BFQ=2 rt. ∠s; ∴ P, F, Q are in the same straight line.

(b) Also ∵ ∠PCQ or ∠ACB$=\frac{1}{3}$ of 2 rt. ∠s, and ∠CPQ or ∠APF$=\frac{1}{3}$ of 2 rt. ∠s, ∴ ∠CQP$=\frac{1}{3}$ of 2 rt. ∠s ∴ △CPQ is equiangular, and hence it is equilateral.

191. *In the figure of I 1, if C and H be the pts of intersection of the ⊙s, and AB be produced to meet one of the ⊙s at K, show that △CHK is an equilateral △.*

For, in △s CBK and HBK; BC=BH, BK is common, and ∠CBK=∠HBK (I. 13), their supplements being equal, (I 32); hence CK=HK (I. 4). So, it can be proved that KC=CH, ∴ △CHK is equilateral

192. If from the ends and the middle pt. of a finite st. line, three ∥ st. lines be drawn meeting an indefinite st. line, the middle parallel is $\frac{1}{2}$ the sum of the two extreme ∥s, when the indefinite line does not meet the finite line; but when it meets it, the middle ∥ is $=\frac{1}{2}$ the difference of the other ∥s

Let AB be the finite st. line, C its middle point, DE the indefinite line · AD, CF and BE the three ∥ st. lines, draw AH ∥ DE meeting CF in G, and BE in H; CF and BE being produced, if necessary.

Case I Since in $\triangle$ ABH, CG is drawn from C the middle point of AB ||l to the side BH, CG is = ½ of BH, and AF and GE are ▭ms, GF = AD or HE, and is ½ (AD + HE), ∴ CF = ½ (AD + BE)

Case II Since CG = ½ BH = ½ (BE + EH), FG = AD or EH = ½ (AD + EH), CG = FG = CF = ½ (BE − AD)

193 In the figure of I 47 —

(1) If BC, CH are joined, these st lines are ||l (2) The points F, A, K, are in one st. line (3) FC and AD are at rt ∠s to one another (4) If GH, KE, FD are joined, the $\triangle$GAH = $\triangle$ABC in all respects, and $\triangle$s FBD, KCE are each = in area to the $\triangle$ ABC

(1) BG bisects the ∠s B and G, ∴ ∠ BGC = ½ a rt ∠. So ∠ACH = ½ rt ∠, ∴ ∠BGC = ∠ACH, and they are alternate ∠s, BG is ||l to CH

(2) Since FA bisects ∠GAB, and AK bisects ∠HAC, ∴ ∠BAF = ½ a rt ∠, and ∠CAK = ½ a rt ∠, ∠FAB + ∠BAC + ∠CAK = ½ a rt ∠ + 1 rt ∠ + ½ a rt ∠ = 2 rt ∠s, ∴ FA and AK are in one st line, points F, A, K are in one st line

(3) Let FC meet AB at P and AD at O. Then in $\triangle$s BPF and OPA, ∠FPB = ∠APO (I 15), ∠PFB = ∠PAO Since $\triangle$BFC is identically = $\triangle$ABD, ∴ ∠FBP = ∠AOP (I 32), but ∠FBP = a rt ∠, ∠AOP is a rt ∠, ∴ FC and AD are at rt. ∠s to each other

(4) From D and E draw DM, EN ⊥ to FB, KC, produced $\triangle$GAH = $\triangle$ABC (I 15 and I 4), ∠ABM = 1 rt ∠ = ∠CBD Take away the common ∠CBM, ∴ ∠ABC = ∠MBD (Ax 3), $\triangle$ABC = $\triangle$BMD (I 26), (for ∠ ABC = ∠ MBD, and ∠ BAC = ∠BMD and BC = BD), BM = BA = BF, $\triangle$ FBD = $\triangle$MBD (I 38) = $\triangle$ ABC So, $\triangle$ KCE = $\triangle$ CNE = $\triangle$ABC, ∴ $\triangle$s GAH, FBD, KCE are each = $\triangle$ ABC

**194 If squares be described on the sides of a rt ∠d $\triangle$, each of the st lines joining the acute ∠s and the opposite ∠s of the square, will cut off from the $\triangle$, an obtuse ∠d $\triangle$ — which will be = that cut off from the square by a st line drawn from the intersection with the side to that ∠ of the square, which is opposite to it.*

From ∠s B, C of the rt ∠d $\triangle$BAC, let st lines BG, CD be drawn to the ∠s of the squares described upon the sides, and from the intersections H and I, let HE, IF be drawn to the opposite ∠s of the squares, $\triangle$BIC = $\triangle$AIF and $\triangle$ CHB = $\triangle$AHE Join AG, AD Now $\triangle$AFI = $\triangle$AIG (I 37), to each of which, add $\triangle$ ABI, ∴ $\triangle$BIF = $\triangle$BAG = $\triangle$BCA (I 37) From each of these equals, take away the $\triangle$BIA, and $\triangle$AIF = $\triangle$AIG = $\triangle$BIC So, it may be proved that $\triangle$CHB = $\triangle$AHE

195. If squares be described on the two sides of a rt ∠d △, the st. line joining each of the acute ∠s of the △ and the opposite ∠ of the square, will meet the ⊥ drawn from the rt. ∠ upon the hypotenuse, in the same point

Let BE, CF be squares described on the sides BA, AC containing the rt ∠ Join DC, BG, they intersect AL, which is ⊥ to BC, in the same point O Produce DE, GF to meet in H Join HA, HB, HC Let BH, CH respectively meet DC, BG in I and K Since EH=AF=AC, and EA=AB, and ∠s HEA, BAC are rt. ∠s, △ HEA = △BAC, and ∠EHA = ∠ BCA = ∠BAL, i e since EH and BA are |||, HAL is a str line, or LA produced passes through H, and HL is ⊥ to BC Again, ∵ AC=CG, AH=BC, and ∠HAC= ∠BCG, ∴ △ HAC= △BCG, ∠ CBK= ∠CHL; but ∠ BCK= ∠ HCL, ∠ BKC = ∠HLC, i e is a rt ∠, and BK is ⊥ to HC It may be shown that CI is ⊥ to BH Hence HL, CI, BK are ⊥s to the sides of the △ HBC, and ∴ they intersect each other in the same point

196. If a △ be described having two of its sides= the diagonals of any quadrilateral, and the included ∠ =either of the ∠s between these diagonals, then the area of the △, is=the area of the quadrilateral

Let the △ABC be described having its sides AB, AC=ED, GF the diagonals of the quad DFEG Then △ABC shall be=the quad DFEG Join BG, BF, then ∵ CF=AG, △CBF=△ABG (I 38) =△ EGD (I 38), and △ABF = △ EDF (I. 38), hence △ABC = the quad DFEG

197 If the sides of the square described upon the hypotenuse of a rt ∠d △, be produced to meet the sides (produced if necessary) of the squares described upon the legs; *they will cut off a △ equiangular and equal to the given △*

Let DB, EC, the sides of the square described on BC the hypotenuse of the rt ∠d △ABC, be produced to meet the sides of sqs described upon BA, AC in K and L; △s BFK, CIL cut off by them are equal and equiangular to △ABC Then ∵ ∠ FBA and ∠KBC are rt ∠s, ∠ FBK = ∠ABC, also, ∠s at F and A are rt ∠s, and FB=BA, ∴ FK=AC, and △FKB = △ABC and equiangular So, it may be proved that △ABC= △LCI and equiangular

198 If from the angular pts of the squares, described upon the sides of a rt ∠d △, ⊥s be let fall upon the hypotenuse produced, they will cut off equal segments, and the ⊥s will together be=to the hypotenuse

Let FM, IN be drawn from the ∠s F, I, of the squares described upon BA, AC, ⊥ to BC (the hypotenuse) produced Then MB shall

be=NC, and FM+IN=BC From A draw AO ⊥ to BC Since ∠FBA is a rt ∠, ∴ ∠FBM + ∠ABO = ∠FBM + ∠BFM, ∴ ∠ABO = ∠BFM, and the ∠s at M and O are rt ∠s, and AB = BF, BM = AO, and FM = BO It may be proved that CN = AO, and IN = CO, ∴ MB = NC, and FM+IN = BO+CO, *i e* = BC

N B The Δ FBM + Δ ICN = Δ ABC

199 Bisect a ▭m, by a st line drawn through a given point.

Let ABCD be the given ▭m, and X be the given point. Let AC and BD be the diagonals of the ▭m Join the *given point* to the *middle point* either of AC or BD and produce the line joining X and the middle pt of either diagonal, to meet two of the ∥ sides. Then this line shall divide the ▭m into two equal areas

200. A ▭m is bisected by any st line which passes through the middle point of one of its diagonals

Let XYZR be a ▭m, M be the middle point of one of its diagonals, YR Through M, draw a line CMD meeting XR in C and YZ in D Now Δ CMR = Δ DMY, (I 29 and I 26) But ΔXYR = ΔZYR, of which parts Δs CMR and DMY are equal, ∴ fig XYMC = fig ZDMR, ∴ XYMC + Δ DMY = ZDMR + Δ CMR, *i e*, fig XYDC=fig ZDCR Hence CD, which passes through the middle point of YR, bisects ▭m XYZR

N. B (1) If through the middle point of any of the diagonals, we drop a ⊥ to one of the sides of the ▭m, and if the ⊥ be produced to meet the opposite side, then the ⊥ shall divide the ▭m into two equal areas

(2) Similarly, if a straight line be drawn ∥ to a given straight line, through the middle point of one of the diagonals, it will divide the ▭m into two equal areas

201 The sum of the distances of any point in the base of an isosceles Δ, from the equal sides is the *same* whatever point in the base is taken.

Let XYZ be an isos Δ From O any point in the base YZ, ⊥s ON and OV are drawn to XZ and XY respectively From Y draw YM ⊥ to XZ Through Y, draw YK ∥ to XZ, meeting NO produced at K The fig YKNM is a ▭m; ∴ YM = KN = KO + ON Again Δ YVO is identically = Δ YKO (I 29), ∴ OV = OK Thus we have YM = KN = KO + ON = VO + ON.

* *202 If CE, BD be the squares, described upon the* side *AC, and the hypotenuse AB, and EB, CD intersect in F, prove that AF bisects the ∠ EFD* (Cam. Ex Pap 1872)

Draw AP, AQ $\perp$ to EB, CD. Then since Δs EAB, CAD are equal, and their bases EB, CD are equal, their altitudes AP, AQ are equal Again $\angle$ AEO = $\angle$ FCO, and $\angle$ EOA = $\angle$ COF, $\therefore$ $\angle$ OFC = $\angle$ OAE = a rt. $\angle$, $\therefore$ fig PAQF is a square, and $\therefore$ AF bisects $\angle$ EFD

203 In fig of I 47, show that (1) $AB^2 + AE^2 = AC^2 + AD^2$; (2) $EK^2 = AB^2 + 4\ AC^2$; (3) $EK^2 + FD^2 = 5\ BC^2$.

(1) Let AL meet BC in O Then $AB^2 = AO^2 + BO^2$ (I 47) $= AO^2 + DL^2$ (for BO = DL, being the opposite sides of the ▭ BL) And $AE^2 = AL^2 + LE^2$ (I 47) $= AL^2 + CO^2$ (for LE = CO, being opposite sides of the ▭ CL), $\therefore$ $AB^2 + AE^2 = AO^2 + DL^2 + AL^2 + CO^2 = (AL^2 + DL^2) + (AO^2 + CO^2) = AD^2 + AC^2$

(2) Produce AC to S, so that CS = CA Join BS Now $\angle$BCS + $\angle$ACB = 2rt $\angle$s, produce EC to M, $\therefore$ $\angle$MCK + $\angle$KCE=2rt. $\angle$s Again $\angle$MCK + $\angle$ACM = a rt $\angle$ = $\angle$ACM + $\angle$ ACB, thus $\angle$ MCK = $\angle$ ACB, and since $\angle$ BCS and $\angle$ ECK are supplementary to equal $\angle$s, $\therefore$ $\angle$BCS = $\angle$ECK, and Δ BCS = Δ ECK $\therefore$ EK = BS, $EK^2 = BS^2 = AB^2 + AS^2 = BA^2 + 4\ AC^2$ (for the square on a line = 4 times the square on its half) (See Ex 233)

(3) From (2) it is evident, that $EK^2 = AB^2 + 4AC^2$, so that $FD^2 = AC^2 + 4\ AB^2$, $\therefore$ $EK^2 + FD^2 = 5\ AB^2 + 5\ AC^2 = 5\,(AB^2 + AC^2) = 5\ BC^2$.

204 Divide an equilateral Δ, into nine equal parts.

Let ABC be an equilateral Δ Trisect AC in D, E, and AD again in F, G (Ex. 98) Draw BF Then AF being $\frac{1}{9}$th of AC, ΔABF is $\frac{1}{9}$th of ΔABC The Δ may be divided in 9 parts equal, in every respect by trisecting all the sides and joining the points of trisection

** 205 To trisect a ▭m by lines drawn —(1) from a given point in one of its sides, (2) from one of its angular points.*

Let ABCD be a ▭m, and P a point in BC Trisect AD in E, F, draw EG, FH ||| to AB Bisect GE in K, draw PKL; join LC, PD, bisect LC in M, and draw MN ||| to PD. Join PM, MD, PN Then BE = $\frac{1}{3}$rd of ABCD, and Δ PKG = Δ KEL (I 26); $\therefore$ ABPL = $\frac{1}{3}$ of ABCD And $\therefore$ LM = MC, Δ PML = Δ PMC, and Δ LMD = Δ MDC; $\therefore$ LPMD = PMDC. Also MN being ||| to PD, the Δ PMD = Δ PND Thus by adding LPD to each, LPND = LPMD = PMDC = Δ PNC. Hence ABP = LPND = PNC and each = $\frac{1}{3}$ of ABCD.

If P is in GH, draw lines through P and pt of bisections of GE. HF

(2) If P coincides with B, L coincides with F, and BFDC can be bisected as PLDC

*206 *Shew that, of all equiangular* ▱ms *of equal* perimeters, *that, which is* equilateral *is the* greatest

Let AC be an equilateral ▱m, and AF equiangular with AD and of the same perimeter Join BD, cutting EF, in O and join EG, cutting BC in P Since AD is a rhombus, ∠ ADB = ∠ABD = ∠ EOB (I 29), CE = EO And AC + AB = EA +AG (hyp), ∴ CE=DG Hence EO=DG Now in the △s EOP, PDG, EO = DG, ∠s at P are equal, and ∠ EOP = ∠PBG (I 29), ∴ they are equal (I 26) Hence △ AEG=AEOP or < ACB, AD>AF

207 *If ABC be an isosceles △, and CD be drawn ⊥ to AD, the sum of the squares on the three sides=AD^2+ 2 BD^2+ 3 CD^2*

Now $AB^2+BC^2=2\ CB^2=2\ CD^2+2\ BD^2$, and $AC^2=CD^2+AD^2$, ∴ $AB^2+BC^2+AC^2=3\ CD^2+2BD^2+AD^2$.

208 On the sides of any △ ABC, equilateral △s BCM, CAN and ABP are described, all *externally* or *towards* the △ Show that AM=BN=CP

In △s PBC and ABM, PB, BC=AB, BM, and ∠ PBC= ∠ABC+$\frac{2}{3}$ rt ∠ = ∠ ABM, PC = AM (I 4) Similarly △ACM = △BCN, AN = BM, ∴ AM = BN = PC

209 If from any point P within a △ ABC, ⊥s PQ, PR and PS are drawn to the sides BC, CA, AB respectively, show that $AS^2 + BQ^2 + CR^2 = AR^2 + CQ^2 + BS^2$

(1) $AS^2+SP^2=AP^2=AR^2+RP^2$ (I 47)

(2) $BQ^2+QP^2=BP^2 = BS^2+SP^2$ (I 47)

(3) $CR^2+RP^2 = CP^2 = CQ^2+PQ^2$ (I 47)

Adding (1), (2) and (3), we have $AS^2+SP^2 + BQ^2+QP^2+CR^2+RP^2 = AR^2 +RP^2+ BS^2+SP^2+ CQ^2+ PQ^2$, ∴ $AS^2+BQ^2+CR^2=AR^2+BS^2+CQ^2$

*210 *To find the* locus *of the vertices of all △s, that have the same base and equal altitudes*

Let MN be the given base, and A the given altitude of a △, to find the *locus* of its vertex Draw MP ⊥ to MN, and make it=A, and through P, draw PQ ∥ to MN, then PQ is the required *locus*. For in PQ take any point V, join MV and NV; then ⊥

from V on MN, would be = MP = A (I 34, Cor), and hence whatever point V is taken in PQ, Δ thus formed has the given base and the given altitude, $\therefore$ PQ is the required *locus* of the vertices

*211 *To find the* locus *of the vertices of all the Δs that have the same base and equal areas.*

Let BC be the given base, and the area of a Δ ABC the given area; to find the locus of the vertices of Δs that have the same base and area Through A draw EF ||| to BC, and it is the required *locus* For in EF take any point D and join BD, CD, then the area of Δ BCD = area of Δ ABC, and it has also the given base BC, since the same be proved wherever the vertex D is taken in the line EF; $\therefore$ it is the required *locus*.

N B If the given area is that of any rectilineal figure, it can be reduced to a ▭m by (I. 45) and ▭m thus found can be applied to *half* of the given base by (I. 44), and then any Δ ABC being constructed on BC, having an altitude = to that of ▭m, will be = given area (I 41).

212 ABC is a Δ, required to draw a str line EF ||| to the base BC, and meeting the other sides in E and F, so that EF may be = BF + CF

Bisect $\angle$s ABC, ACB by BD and CD meeting at D. Now $\angle$ DBC + $\angle$ DCB are < 2 rt $\angle$s (I 17) Through D draw EDF ||| to BC (I 31), $\therefore$ $\angle$ EDB = $\angle$ DBC (I 29) and $\angle$ DBC = $\angle$ DBE (const); $\therefore$ $\angle$ DBE = $\angle$ EDB (Ax. I), $\therefore$ EB = ED (I 6). So it may be shown that DF = FC, $\therefore$ EB + FC = ED + DF = EF

213. Draw a line ||| to the base BC of any Δ ABC meeting the other sides in D and E, so that DE may be = the difference of BD and CE

Produce BC to M, bisect $\angle$s ABC and ACM by BP and CP meeting at P Through P, draw PED ||| to BC (I 31) meeting AC in E and AB in D Now $\angle$ DPB = $\angle$ PBC (I 29) and $\angle$ PBC = $\angle$ PBD; $\therefore$ $\angle$ DPB = $\angle$ DBP, $\therefore$ DP = DB, $\angle$ EPC = $\angle$ PCF (I. 29), and $\angle$ PCF = $\angle$ PCE (const), $\angle$ EPC = $\angle$ PCE, $\therefore$ EP = EC, $\therefore$ BD − CE = DP − PE = DE

214 If the three sides of one Δ be respectively $\perp$s to those of another Δ, the Δs are equiangular.

Let the sides of the Δ DEF be $\perp$s to the sides of Δ ABC, it is required to prove that Δs DEF, ABC are equiangular Since $\angle$s CHE, EGC are rt $\angle$s, $\angle$ HCG + $\angle$ HEG = 2 rt. $\angle$s (I 32 Cor), and $\angle$ HED + $\angle$ HEG = 2 rt. $\angle$s Take away the common $\angle$ HEG, and we have $\angle$ HCG = $\angle$ DEF, *i e* $\angle$ ACB = $\angle$ DEF. So $\angle$ BAC = $\angle$ EFD, and $\angle$ ABC = $\angle$ EDF

215 If through an ∠ of a ▱m, any str. line is drawn, the ⊥ drawn to it from the opposite ∠, is = the *sum* or *difference* of the ⊥s drawn to it, from the two remaining ∠s, according as the given str line falls *without* the ▱m, or *intersects* it

Case I When the given line falls without *the* ▱m

Let ABCD be a ▱m Through A draw any st line MAN and let CE, BN and DM be ⊥s on this line Through C draw any str line RCS ||l to MN meeting NB produced in S, and MD produced in R Produce SR to meet AD produced Q, in the Δs ADM and BSC, ∠ AMD = ∠BSC = a rt ∠ and ∠MAD = ∠AQC (I 29) = ∠BCS, and AD = BC, ∴ Δ ADM is identically = ΔBSC, ∴ BS = MD, ∴ DM + BN = BN + BS = NS, but NS = CE (I. 34), ∴ DM + BN = CE

Case II When the given st line intersects a side of the ▱m

Let ABCD be a ▱m, and let AE be drawn from ∠A to intersect CD Draw CE ⊥ to AE from ∠ C (the ∠ opposite to ∠A), BH and CL ⊥s from B and D to AE, are also drawn It is required to prove that CE = BH − DL Through C draw CM ||l to AE, meeting DL produced at G, and BH at K In the two Δs ABH and DCG, ∠AHB = ∠DGC (being rt ∠s), ∠DCG = ∠CMB = ∠BAH (I 29), and AB = DC, ∴ BH = DG (I 26), but HK = LG (I 34), ∴ the remainders BK = DL, ∴ BH − BK = HK = GL = CE or BH − DL = CE

216 From the angular pts of a ▱m, ⊥s are drawn to any str line which is without the ▱m Show that the sum of the ⊥s drawn from one pair of opposite ∠s, is = the sum of those, drawn from the other pair

Let ABCD be a ▱m, and let PQ be any st line without it Let the diagonals AC and BD intersect each other at O Let AM and CN be ⊥s to PQ, from one pair of opposite ∠s, and BP and DQ ⊥s on PQ from the other pair of opposite ∠s, and OE be the ⊥ from O to PQ AM + CN = 2 OE (Ex 191) = BP + DQ

227 AB is given st line bisected at O, and AX, BY are ⊥s drawn from A and B on any other st line show that OX = OY

Through O draw OMP ||l to AX or BY cutting XB at M In the Δ ABX, ∵ O is the middle point of AB and OM is ||l to AX, ∴ M is the middle point of BX (Ex 1 P 96, Text) Again in ΔXBY, ∵ M is the middle pt of XB and MP is ||l to BY, ∴ P is the middle pt of XY, (Ex. 1 P 96, Text), ∴ XP = PY In Δs OXP and OYP, XP = PY, and OP is common to both, ∠XPO = ∠YPO, for ∠AXP + ∠OPX = 2 rt ∠s (I. 29), and ∠AXP = a rt. ∠,

∠, ∴ ∠XPO = a rt. ∠, ∴ ∠OPY = a rt. ∠ (I. 13), ∴ OX = OY (I. 4)

218 From the extremities of the base of a Δ, ⊥s are drawn to the opposite sides (produced if necessary); show that the str. lines which join the middle point of the base to the feet of the ⊥s, are equal.

Let ABC be a Δ, BE and CD are ⊥s to AC and AB, F is the middle point of the base BC. In Δ BEC, ∠ BEC is a rt ∠ F is the middle point of the hypotenuse, ∴ FE is ½ of BC, similarly from Δ DBC, DF is ½ of BC, (Ex 107), ∴ FE = DF.

219 If in a diagonal of a ▭m, any two points equidistant from its extremities, be joined to the opposite ∠s, the figure thus formed, will also be a ▭m

In the Δs ADF and CBE, AD = BC, DF = BE (hyp.), ∠ADF = ∠ CBE, ∴ these 2 Δs are equal in all respects (I 4), ∴ AF = EC, ∠ AFE = ∠ FAD + ∠ ADF (I 32), ∠CEF = ∠CBE + ∠BCE, (I 32) But ∠FAD + ∠ADF = ∠ CBE + ∠BCE, ∴ ∠ AFE = ∠ CEF, and they are alternate ∠s, ∴ AF is ||l to EC, but it has been shown that, these two sides are equal, the fig AECF is a ▭m (I 33)

220 ABC is a given equilateral Δ, and in the sides BC, CA, AB, the pts X, Y, Z, are taken respectively, so that BX = CY = AZ; AX, BY, CZ are drawn, intersecting in P, Q, R Prove that the ΔPQR is equilateral

In the Δs, ABX and BCY, AB = BC, BX = CY and ∠ABX = ∠BCY, ∴ AX = BY and Δ ABX is identically = ΔBCY (I 4) So, it may be shown that AX = BY = CZ From Δs ABX and BCY, it may be shown that ∠BAX = ∠CBY, and from Δs ABX and CAZ, it may be shown that ∠AXB = ∠CZA. Now in the Δs AZP and BXQ, BAX = ∠CBY, ∠AXB = ∠CZA, and AZ = BX ∴ AP = XQ (I 26), similarly BQ = RY and ZP = RC From the equals, AX, BY and CZ take away the equals, ∴ PQ = QR = PR ∴ ΔPQR is equilateral

***221 Construct a Δ, having given its three *medians*.**

Let M, N, Q, be the three *medians* Take ⅔ of each of the str lines M, N, Q, and with these construct the Δ BGH so that BG, GH, HB are respectively = ⅔ of M, N, Q, (I. 22) Draw GC ||l to BH and HC ||l to BG meeting GC at C Join BC cutting GH at D. Produce HG to A, making GA = HG Join BA, CA Then ABC is the Δ required Produce BG to meet AC at E, and CG to meet AB at F Now, since BGCH is a ▭m (const.), and since the diagonals of a ▭m bisect each other, D is the middle pt. of BC Again, since GF passes through the middle pt of AH and is ||l to HB, ∴ F is the middle point of AB. So E is the middle point of AC. (Ex I. p. 96, Text.) Thus, AD,

BE, CF are the bisectors of the sides of the Δ, and $\therefore$ by (Ex 4 and Cor. P 105, Text) AG=$\frac{2}{3}$ AD, BG=$\frac{2}{3}$ BE, CG=$\frac{2}{3}$ CF. But AG=GH=$\frac{2}{3}$ of N, BG=$\frac{2}{3}$ of M, and CG=BH=$\frac{2}{3}$ of Q, AD, BE, CF are respectively=the str. lines M,N, Q. Thus Δ ABC has the bisectors of its sides=the given str lines, and is the Δ required

*222 *A pt is taken within a square, and str lines drawn from it to the angular points of the square, and* $\perp$*s to the sides; the squares on the first, are double the sum of the squares on the last Shew that these* sums *are* least, *when the point is in the* centre *of the square*

Let E be the pt in the square ABCD, and EA, EB, EC, ED be drawn to the angular points, and EG, EH, EK, EL be drawn $\perp$s to the sides, of which two and two will be in one and the same straight line Then $BE^2 = LE^2+BL^2 = LE^2+FG^2$ Similarly expressing EC^2, ED^2, EA^2 in terms of GE^2, HE^2, KE^2, LE^2, the required result is obtained When (F, the intersection of the diagonals AC, BD) is the pt taken, draw PFQ $\perp$ to AD or BC. Join AO, (O being the intersection of BD, LH) Then it is evident that $\angle$AFB is a rt $\angle$, and $\angle$ LBO and $\angle$ BOL are each=$\frac{1}{2}$ a rt $\angle$, $\because$ AO>AF, and LO=LB or EG Thus EG^2+EK^2 or $LO^2+LA^2 = AO^2$, that is $> AF^2$ or $> (FP^2+FQ^2)$ Hence these sums are *least*, when F is the point taken

*223 *Inscribe a rhombus within a given* ▭*m, so that, one of the angular points of the rhombus, may be at a given point in a side of the* ▭m

Let ABCD be a ▭m, and P a given point in the side AB Join BD, AC, and through O their pt of intersection, draw POQ, cutting CD at Q Also, through the same pt draw SOR at rt $\angle$s to POQ, meeting the sides, produced if necessary, at S, R. Join SP, PR, RQ, QS, then PRQS shall be the rhombus required. For in Δs POB and DOQ, $\angle$PBO=$\angle$ODQ (I 29), and $\angle$POB=$\angle$ DOQ (I 15) also OB=OD (Ex 128), hence OP=OQ. (I 26) So it may be proved that, SO=OR Now since PO and SR bisect each other at rt $\angle$s, SP=PR=RQ=QS (Ex 8), and SPQR is a rhombus

224 **If two Δs of equal area are on opposite sides of the same base, the joining st line of their vertices is bisected by the base, and conversely**

Let Δs ABC, DBC be of equal area and on opposite sides of the same base BC Draw BP, CP ||ls to CD, BD respectively, and on the same base of BC as Δ ABC Join AP Then BPCD is a ▭m, Δ BPC=Δ DBC=Δ ABC; $\therefore$ AP is ||l to BC Also, Y being the intersection of diagonals BC, PD of ▭m, DY=YP. And since XY is ||l to AP, X is mid pt of AD

For the *Converse*, the same construction being made as before. Y is middle point of DP; ∴ if X is middle point of AD, then XY is ||| to AP, hence Δ ABC = Δ PBC = Δ DBC

225 Construct a Δ = to the sum or difference of two given Δs

Let M and N be the two given Δs If they are not on equal bases, construct a Δ ABC = N, having the base AB = base of M (Ex 2, P. 111, Text). From A draw AL ⊥ to AB From AL cut off AD = altitude of the Δ ABC, and cut off AF = altitude of Δ M, and from FL cut off FG = AD Join GB, FB and DB Draw FE and DC ||| to AB Join AE and EB Δ AFB + Δ FGB or Δ AGB = Δ EAB + Δ DAB = Δ EAB + Δ ABC = Δ M + Δ N

226 In a rt ∠d Δ, four times the sum of the squares on the medians, which bisect the sides containing the rt ∠ = 5 times the square on the hypotenuse.

Let ABC be a Δ, rt. ∠d. at A, and let BE and CD be the medians from B and C. $4BE^2 = 4(AB^2 + AE^2)$ (I 47); $4DC^2 = 4(AD^2 + AC^2)$; ∴ $4(BE^2 + DC^2) = 4(AB^2 + AC^2 + AD^2 + AE^2) = 4(BC^2 + DE^2) = 4BC^2 + BC^2$ (for $2DE = BC$) $= 5BC^2$.

227. Three times the square on a side of an equilateral Δ = four times the square on the ⊥ drawn from any vertex to the opposite side

Let ABC be an equilateral Δ, AD the ⊥ on BC, evidently AD bisects BC $AB^2 = AD^2 + BD^2$ (I 47), $AD^2 + \frac{1}{4}BC^2 = AD^2 + \frac{1}{4}AB^2$; ∴ $\frac{3}{4}AB^2 = AD^2$ or $3AB^2 = 4AD^2$.

228 The equilateral Δ described on the hypotenuse of any rt. ∠d Δ, is = the sum of the equilateral Δs described on the sides.

Let BAC be a rt ∠d Δ, having ∠ A = a rt ∠ and BDC, CEA, AFB are the equilateral Δs on its sides It is required to prove that Δ BDC = Δ CEA + Δ AFB Draw FG ⊥ to AB, then evidently G is the middle pt of AB, and FG ||| to AC; ∴ Δ FGC = Δ FGA (I 37) = ½ Δ AFB, ∴ Δ FGC + Δ FGB = Δ AFB Adding to each of these equals Δ BGC, the whole Δ FBC = Δ AFB + Δ BGC = Δ AFB + ½ Δ ABC (∵ G is the middle point of AB) .. .. (1) Now draw AH, DK ⊥s to BC join AK, AD, HD Then ∵ AH is ||| to DK, Δ AHD = Δ AHK (I 37) Hence Δ ABD = Δ BHD + Δ ABK = Δ BHD + ½ Δ ABC (∵ K is the middle point of BC) .. (2) But Δ FBC = Δ ABD, since FB = BA, BC = BD, and ∠ FBC = ∠ ABD. Hence from (1) and (2), Δ AFB = Δ BHD. Similarly it may be shown that Δ AEC = Δ CHD, ∴ Δ AFB + Δ AEC = whole Δ BDC

229 A Δ is = to the *sum of difference* of two Δs on the same base (or on equal bases), if the altitude of the former is = to the sum or difference of the altitudes of the latter.

Let BAC and BDC be two Δs on the same base BC, Δ BAC having the greater altitude. Draw BM $\perp$ to BC. Through A and D draw AE and DF ||l BC meeting BM at E and F. From EM cut off EN=FB, and join NC, EC and FC. NB is the sum of the altitudes of Δs BAC, BDC. And since EN=FB; $\because$ Δ NEC=Δ FBC (I. 38), $\because$ Δ BAC + ΔBDC = Δ BEC+ Δ FBC=Δ BEC+Δ NEC=Δ NBC.

If through N a line be drawn ||l to BC, and if B and C be joined to any point in the ||l line so drawn, then each new formed Δ=Δ NBC, having NB for their altitude. ΔNEC=Δ FBC= Δ BDC (I. 38 and 37), and Δ EBC=ΔABC (I. 37), $\because$ ΔNBC =Δ EBC or Δ ABC=ΔNEC=Δ FBC=ΔBDC. Thus ΔBDC is the difference of two Δs NBC and ABC. The altitude of ΔDBC=the difference of the altitutes of Δs NBC and ABC.

230. ABC is a rt. $\angle$d Δ, AD (the $\perp$ from A upon the hypotenuse BC) is produced in the direction DA till it meets a side produced of the square on AC in O. Prove that (1) it will meet a side produced of the square on AB in the same pt. O, (2) that AO shall be=BC, (3) and that if O be joined with B, and A with E, (the extremity of the side BE of the square on BC), the fig. OAEB shall be a ▭m=in area to AB^2. *(Cal. Ex. Pap. 1871.)*

Let AFGC and AKHB be the sqs. on AC and AB. Join OK, $\therefore$ $\angle$ABC+$\angle$ ACB=a rt. $\angle$=$\angle$BAD+$\angle$ABD (I. 32), $\therefore$ $\angle$BAD =$\angle$ ACB, also $\angle$ BAD=$\angle$OAF (I. 15), $\therefore$ $\angle$ OAF= $\angle$ ACB, (Ax. I.) also $\angle$BAC=$\angle$OFA, a rt. $\angle$, and AC=AF, $\therefore$ OA=BC (I. 26) and OF=AB=AK. Since $\angle$s KAF and AFO are rt. $\angle$s; $\because$ AK ||l to OF (I. 28), $\therefore$ OK=and ||l to AF (I. 33) and $\angle$ AKO is a rt. $\angle$ (I. 29), also $\angle$ AKH is a rt. $\angle$, $\therefore$ HK is in the same straight line with KO (I. 14), wherefore HK produced, will pass through the point O. Again $\angle$ODB=$\angle$DBE=a rt. $\angle$, $\therefore$ OD ||l to BE (I. 27), also AO=BC=BE, $\therefore$ BO=and ||l to EA (I. 33), $\therefore$ OAEB is a ▱m, and ▱m OAEB=2 Δ ABO (I. 34)= ▱m ABHK, (I. 41)=the square on AB.

**231. OKBM and OLDN are ▱ms about the diagonal of a ▱m ABCD. In MN, which is ||l to BA, take any pt. P and prove that, if PC, produced if necessary, meet KL in Q, BP will be ||l to DQ.* (Cam. Ex. Pap. 1868.)

Through Q draw RQS ||l to NM. Then Δ BQD=Δ ODQ+ ΔOQB=½▱NQ+½▭OS (I. 41)=½▭ NRSM. And Δ PQD= ΔPDC$-$$\Delta$ QDC=½▭NC$-$½▭RC (I. 41)=½▭NRSM, $\therefore$ Δ BQD=Δ PQD, $\therefore$ PB is ||l to QD (I. 39).

BOOK II.

232. If two st lines AB and BC be each of them divided into any number of parts AD, DE, EB and BF, FG, GC, the rect under two lines is = in area to the sum of all rects under all the parts of the one, taken separately with all the parts of the other.

AB BF = BF AD + BF DE + BF EB, AB FG = FG AD + FG DE + FG EB, AB GC = GC AD + GC DE + GC EB, but AB BC = AB.BF + AB FG + AB GC ∴ it is equal in area to the sum of all the rects. under all the parts of the one, taken separately with all the parts of the other

233 The square on a straight line, is = *four* times the square on half the line

On AB describe a sq ABCD (I 46) Bisect AB, AD in the points E and F (I 10) Draw EG ||| to AD, and FH ||| to AB. Then the sq ABCD is divided into *four* squares, each of which is $=AE^2$, ∴ $AB^2=4\ AE^2$

234 If four points A,B,C,D, *are in order* on the same st line, then prove AC BD = AB CD + BC AD (Euler)

Applying (II 1) repeatedly, we have AC BD = (AB + BC) BD = AB BD + BC BD = AB (CD + BC) + BC BD = AB CD + BC AB + BC BD = AB. CD + BC (AB + BD) = AB CD + BC AD

235 In a Δ, whose vertical ∠ is a rt ∠, a straight line is drawn from the vertex ⊥ to the base Show that the rectangle contained by the segments of the base = the square on the ⊥r (*Bombay Ex Pap, 1876*)

Let ABC be the Δ, and BAC the rt ∠. Draw AD ⊥ to BC Then $BD^2+CD^2+2BD\ DC=BC^2$ (II 4) $=BA^2+CA^2=(BD^2+DA^2)+(DA^2+CD^2)$, ∴ $2BD\ DC=2DA^2$ ∴ rect BD CD $=DA^2$.

236 If a ⊥ CD be drawn from the vertex of a scalene ΔABC, the difference of the squares on the sides AC and CB = in area to twice the rect under the base AB and the distance DE of its middle pt from the ⊥ (Cf Ex 8, p 145, Text).

237. In a rt. ∠d Δ, if a ⊥ be drawn from the rt ∠ to the hypotenuse the square on either side forming rt ∠ = the rect contained by the hypotenuse and the segment of it adjacent that side

Let ABC be a rt $\angle$d Δ, to prove $AB^2=CB.BD$, or $AC^2=BC.CD$ From A draw a $\perp$ to BC. Now $CB.BD=BD^2+BD.DC$ (II 3)$=BD^2+AD^2$ for $BD.DC=AD^2$ (Ex 235), $\therefore CB.BD=BD^2+AD^2=AB^2$, so shall $CA^2=BC.CD$

238 Prove II 9 from II 4, II 5 and Ex 233

See fig (II. 9), $AB^2=AQ^2+QB^2+2AQ.QB$ (II 4) But $AP=\frac{1}{2}AB$ or $2AP=AB$, $\therefore AB^2=4AP^2$ And $AQ.QB=AP^2-PQ^2$ (II 5 Cor), $4AP^2=AQ^2+QB^2+2AP-2PQ^2$, or $AQ^2+BQ^2=2AP^2+2PQ^2$

239 If from one of the base $\angle$s of an isosceles Δ, a $\perp$ is drawn to the opp side, then twice the rectangle contained by that side and the segment adjacent to the base = the square on the base

Let ABC be an isosceles Δ, BE is the $\perp$ drawn from B one of the base $\angle$s, to the opposite side AC To prove $2AC.CE=BC^2$

$\angle$ ACB is = an *acute* $\angle$, $AB^2=BC^2+AC^2-2AC.CE$ (II 13) or $AB^2+2AC.CE=BC^2+AC^2$, but $AB=AC$, $AB^2=AC^2$, $\therefore 2AC.CE=BC^2$

240 A straight line is divided into two parts, show that if twice the rect of the parts is = the sum of the squares described on the parts, the st line is bisected

Let the st line AB be divided at C so that $2AC.CB=AC^2+BC^2$ Then C shall be the mid pt of AB If not, let D be the mid pt, then $\because AB^2=AC^2+CB^2+2AC.CB$ (II 4) $=4AC.CB$ (Hyp), and $AB^2=4BD^2$ (Ex 233); hence $AC.CB=BD^2$, which is absurd (II 5), $\therefore$ D is not the mid pt of AB So it may be proved that no other point but C can be the mid pt of AB Hence C is the mid pt of AB

241 Divide a given str line into two parts, such that the rect. contained by them shall be the **greatest** *possible* (**Bombay Ex. Pap 1874, Cal Ex. Pap 1872**)

Let AB be the give str line Bisect it in C, and in it take any other pt D Then AC.CB shall be > AD.DB For $AD.DB+CD^2=CB^2$ (II 5)$=AC.CB$, *i e* the rect AC.CB > AD.DB by CD^2 So AC.CB may be proved to be > any other rectangle Hence AC.CB is the *greatest*

242 Divide a given st line into two parts, such that the sum of the squares on the two parts may be the **least** *possible* (**Bombay Ex. Pap 1869, Cal Ex Pap 1882**).

Let AB be the given straight line. Bisect it in C, and in it take any other point D Then AC^2+CB^2 shall be $<AD^2+DB^2$.

For $AD^2+DB^2=2AC^2+2CD^2$ (II 9) $=AC^2+CB^2+2CD^2$, i e $AC^2+CB^2<AD^2+DB^2$ by $2CD^2$. So it may be proved that $AC^2+CB^2<$ the squares on any two parts into which AB may be divided Hence AC^2+CB^2 is the *least*

243. Any rectangle is $\frac{1}{2}$ of the rect contained by the diagonals of the squares on its two sides

Let XPQY be a rectangle Describe squares *(external)* PNSQ and PMRX on PQ and PX respectively (I 46), join PS and PR It is required to prove that $XPQY=\frac{1}{2}$ RP PS Since RP is a diagonal of the square on PX, $\therefore RP^2=2RX^2$ So $PS^2=2PQ^2=2XY^2$; and $\angle RPX=\frac{1}{2}$ a rt $\angle=\angle QPS$. and $\angle XPQ=$ a rt $\angle$, .. RP and PS are in one st line (I 14), and in $\triangle RYS$, $\because \angle YRP=\frac{1}{2}$ a rt. $\angle=\angle YSP$, $\therefore RY=SY$, $\therefore RS^2=2RY^2$ Now (1) $RS^2=RP^2+PS^2+2RP\ PS$ (II 4), and (2) $RY^2=RX^2+XY^2+2RX\ XY$ (II 4) From (1) and (2), we have $RS^2=2RY^2=$ $2RX^2+2XY^2+4RX\ XY=RP^2+PS^2+2RP\ PS$ (Rejecting the equals) we have, $4RX\ XY=2RP\ PS$, $RP\ PS=2RX\ XY=2PX\ XY$ $=2XPQY$, $\therefore$ rect $XPQY=\frac{1}{2}RP\ PS$

244 If a st line CD be drawn from the vertex of an isosceles $\triangle ABC$, *to any point in the base or the base produced*, the rect under the segments of the base AD and DB, is = in area to the difference between the square on this line CD and the square on either side AD or BC of the $\triangle ABC$. *(Bombay Ex Pap 1882)*

Bisect the base AB in E (I 10), and draw CE a $\perp$ on AB When D is in AB,—see the proof in (*a*), when D is without AB,—see the proof in (*b*)

(*a*) $AD.\ DB+DE^2=AE^2$ (II 5), $\therefore AD\ DB=AE^2-DE^2$ $=(AC^2-CE^2)-(CD^2-CE^2)=AC^2-CD^2$

(*b*) $AD.DB+AE^2=ED^2$ (II 6), $\therefore AD\ DB=ED^2-AE^2=$ $(CD^2-CE^2)-(AC^2-CE^2)=CD^2-AC^2$

***245. In any $\triangle$, if a line be drawn from the vertex bisecting the base, the sum of the squares on two sides of the $\triangle$, is double the sum of the squares on the bisecting line and on half the base.** *(Bombay Ex Pap 1863; Cal. Ex Pap. 1868)*

From the vertex A of the $\triangle ABC$, let AD be drawn to the pt of bisection of the base, AB^2+AC^2 shall be $=2AD^2+2DB^2$. From A draw AE $\perp$ to BC, then $AB^2=AD^2+DB^2+2BD\ DE$ (II 12), and $AC^2=AD^2+DC^2-2CD\ DE$ (II 13) $=AD^2+DB^2-2BD\ DE$, hence $AB^2+AC^2=2AD^2+2DB^2$ **(This Exercise is very important)** *(See Text, Ex. 24, p. 147)*

246 **Apollonius's theorem.** *If ABC be any* $\triangle$, *and D any pt the base BC so that m BD=n DC, then shew that* $m\ AB^2+n\ AC^2=(m+n)\ AD^2+m\ BD^2+n\ DC^2$

Draw AP $\perp$ to BC, then (1) $AB^2=AD^2+DB^2+2BD\ DP$ (II 12), $AC^2=AD^2+DC^2-2CD\ DP$ (II 13), (2) Multiply the *first* equation by m and the *second* by n and add, thus we get $m\ AB^2+n\ AC^2=(m+n)\ AD^2+m\ DB^2+n\ DC^2$, since $2m\ BD\ DP=2n\ CD\ DP$ (Hyp) (*Ex 245, is a special case of 246*)

247 In an isosceles $\triangle$, if a $\perp$ be drawn from one of the $\angle$s at the base to the opp side, show that the square on the $\perp$ = twice the rect obtained by the segments of that side together with the square on the segment adjacent to the base

Let ABC be an isosceles $\triangle$, having its vertex at A Draw BD $\perp$ to AC Now $AC^2=AD^2+CD^2+2\ AD\ DC$ (II 4), but $AB^2=AC^2$, AB^2 or $(AD^2+BD^2)=AD^2+CD^2+2AD\ DC$ (from above), $\therefore BD^2=CD^2+2AD\ DC$

248 (In II 11) Show that the squares on the whole line and one of the parts, are=thrice *the square on the other part*

$AB^2+BH^2=2AB\ BH+AH^2$ (II 7) $=2AH^2+AH^2=3AH^2$, *for* $AB\ BH=AH^2$ (II 11)

249 In the fig of Euc (II 11), if CH be produced to meet BF at L, show that CL is $\perp$ to BF

The two $\triangle$s FAB, HAC are equal in all respects (I 4) Hence $\angle ACH=\angle FBA$ In the two $\triangle$s AHC, LHB, $\angle ACH=\angle LBH$ and also $\angle AHC=\angle LHB$, . the remaining $\angle HAC=\angle HLB$ (I 32), hence $\angle HLB$=a rt $\angle$, *i e* CL is $\perp$ to FB

250 In the fig II 11, prove that the st lines GB, DF, AK are ||l

Join FK, AD Then $\triangle\ AFK=\frac{1}{2}AH^2$ (I 41), and for the same reason $\triangle\ ADK=\frac{1}{2}$ the rectangle HD But sq on AH= rect HD, $\therefore \triangle AFK=\triangle ADK$ Hence FD is ||l AK (I 39) Join AG, BK, $\triangle\ AGK=\frac{1}{2}$ rect FK and $\triangle\ ABK=\frac{1}{2}AD^2$ (I. 41). But the rect $FC=AD^2$, $\therefore \triangle AGK=\triangle ABK$ Hence GB is ||l AK (I 39) Thus GB is ||l FD (I 30), GB, FD, AK are ||l to one another

251. The sum of the squares on the sides of ▱m, is=the sum of the squares on the diagonals. (*Cal Ex. Pap 1859, Bom Ex Pap 1866*)

Let ABCD be ▱m To prove that $AC^2+BD^2=AD^2+AB^2+BC^2+CD^2$ Draw CE ||l to BD. Produce AD to meet CE in E;

$AD=BC$ (I 34) $=DE$. Hence $AC^2+BD^2=AC^2+CE^2=2AD^2+2DC^2$ (Ex. 245) $=(AD^2+BC^2)+(AB^2+CD^2)$.

252 **If a st line be drawn through one of the ∠s of an equilateral Δ, to meet the opp' side produced, so that the rectangle contained by the segments of the base is = to the square on the side of the Δ, show that the square on the st line so drawn is double of the square on a side of the Δ.**

Let ABC be an equilateral Δ Through ∠C a st line CD is drawn to meet the opp side AB produced at D, and it is given that rec. AD DB $=CB^2$, to prove $CD^2=2CB^2$

AD DB $+EB^2=ED^2$ (II 6), adding CE^2 to each; AD DB $+(EB^2+CE^2)=ED^2+CE^2$ or AD DB $+BC^2=CD^2$ (I 47), *i e* $BC^2+BC^2=CD^2$ (hyp), ∴ $2BC^2=CD^2$

*253 *Produce a given straight line so that the rect contained by the whole line thus produced and the part of it produced, shall be = a given square* (Cal Ex Pap 1865)

Let AB be the given straight line Bisect it in the point C (I. 10) At B draw BD ⊥ AB and = a side of the given square Join CD Produce AB to E making CE = CD Then E is the pt required, ∵ AE EB $+CB^2=CE^2$ (II 6) $=CD^2=CB^2+BD^2$ (I. 47), ∴ AE EB $=BD^2=$ the given square

254. **If two sides of a trapezium be ∥l to each other, the squares on its diagonals are together = to the squares on its two sides which not ∥l and twice the rectangle contained by its ∥l sides**

Let the sides AB, DC of the trapezium ABCD be parallel draw the diagonals AC, BD. Then shall AC^2+BD^2 be $=AD^2+BC^2+2AB\ DC$ Let fall the ⊥s CE, DF. Then $DB^2=DA^2+AB^2+2AB\ AF$ (II 12), and $AC^2=CB^2+AB^2+2AB\ BE$, (II 12), hence $AC^2+DB^2=AD^2+CB^2+2AB^2+2AB\ BE+2AB\ AF$ Now $AB\ FE=AB\ FA+AB\ AB+AB\ BE$, (II 1), ∴ $AC^2+DB^2=AD^2+CB^2+2AB\ DC$.

255 **In any Δ, if a ⊥ be drawn from the vertical ∠ to the base, the sum of the squares on the sides forming that ∠, together with twice the rect contained by the segments of the base, is = to the square on the base, together with twice the square on the ⊥.**

Let ABC be any Δ, AD the ⊥ from the vertical ∠A to the base. To prove $AB^2+AC^2+2BD\ DC=BC^2+2AD^2$ Now $AC^2=AD^2+DC^2$ (I 47). $AB^2=AD^2+BD^2$ (I 47), ∴ $AC^2+AB^2=2AD^2+(DC^2+BD^2)$, and $BC^2=BD^2+CD^2+2BD\ DC$ (II. 4),

$BD^2 + CD^2 = BC^2 - 2BD\ DC$, $\therefore$ $AC^2 + AB^2 = 2AD^2 + (BC^2 - 2BD\ DC)$, $\therefore$ $AC^2 + AB^2 + 2BD\ DC = BC^2 + 2AD^2$

256 Divide a given st line into two parts, so that the rectangle contained by them, shall be = to the square on a given st line, which is less than ½ the line to be divided (*Cal Ex Pap 1863*)

Let AB be the given st line to be divided Draw AD at rt ∠s to AB, and = a side of the given square Describe a semi ◯ upon AB as diameter From D draw DE ∥ AB, cutting the semi ◯ in E From E draw EF ⊥ AB Then F is the required point of division Let C be the centre, and join CE Since $AF\ FB + CF^2 = CB^2$ (II 5) $= CE^2 = CF^2 + EF^2$ (I 47), $\therefore$ $AF\ FB = EF^2 = AD^2 =$ the given square

257 If a st line AB be *divided in medial section* at H, and if M be the mid pt of the greater segment AH, prove that the △ whose sides are AH, MH, BM respectively, must be rt ∠d or $AH^2 + MH^2 = BM^2$

Since M is the mid pt of AH, and AH is produced to B, $\therefore$ rect $AB.\ BH + MH^2 = BM^2$ (II 6), but $AB\ BH = AH^2$ (hyp), $\therefore$ $AH^2 + MH^2 = BM^2$

258 In a △ APB, $AP^2 < PB^2$ by a constant quantity. *Prove that P must be on a certain straight line* (Cal Ex Pap 1872)

From P draw PD ⊥ AB Then $PB^2 = AP^2 + AB^2 - 2AB,\ AD$ (II 13) Take AP^2 from both these equals, $PB^2 - AP^2 = AB^2 - 2AB\ AD$ Let Q be some other position of P If QR be drawn ⊥ AB, then we can prove that $QB^2 - AQ^2 = AB^2 - 2AB\ AR$, $\therefore$ $AB^2 - 2AB\ AD = AB^2 - 2AB\ AR$, being each of them = *a constant quantity*, then AD = AR, $\therefore$ QR and PD will coincide Hence the position of P will always remain on the st line drawn from D ⊥ AB

259 The sum of the squares on the sides of a quadrilateral, is greater than the sum of the squares on its diagonals by four times the square on the st line which joins the mid points of the diagonals *(Cam Ex. Pap 1870.)*

Let ABCD be a quadrilateral, and let O and P be the mid points of its diagonals BD and AC. Join OP, BP, DP To prove $AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2 + 4OP^2$

In △ ABC, $AB^2 + BC^2 = 2BP^2 + 2CP^2$ (Ex 245), $\because$ $AP = CP = \frac{1}{2}AC$ In △ ADC, $CD^2 + DA^2 = 2DP^2 + 2CP^2$ Adding we get, $AB^2 + BC^2 + CD^2 + DA^2 = 4CP^2 + 2BP^2 + 2DP^2 = 4CP^2 +$

$2(BP^2+DP^2)$; but in Δ BPD, $\because$ O is the middle point of BD, $\therefore$ $BP^2 + DP^2 = 2BO^2 + 2OP^2$ (Ex 245), $\therefore$ $2(BD^2 + DP^2) = 4(BO^2+OP^2)$, $\therefore$ $AB^2+BC^2+CD^2+DA^2 = 4CP^2+2(BP^2+PD^2) = 4\{CP^2+(BO^2+OP^2)\} = 4CP^2+4BO^2+4OP^2 = 4(\frac{1}{2}AC)^2+4(\frac{1}{2}BD)^2 + 4OP^2 = AC^2+BD^2+4OP^2$.

260 In any quadrilateral, the squares on the diagonals are together = to twice the sum of the squares on the st. lines joining the mid points of opposite sides *(Cam Ex Pap. 1859)*

Let ABCD be a quadrilateral; and E, F, G, H the mid. pts of its sides Then, EF joins the mid pts of BA, BC, $\therefore$ it is ||l AC (Text, p 96, Ex 2), and AC=2EF (Text, p 97, Ex 3) So HG is ||l AC, and EH, FG are ||l BD, EFGH is a ▱m $AC^2 = 4EF^2$ (Ex. 233), $BD^2 = 4FG^2$ (Ex 233) Adding $AC^2 + BD^2 = 4\,EF^2 + 4\,FG^2 = 2(2\,EF^2 + 2\,FG^2) = 2\{(EF^2 + GH^2) + (FG^2 + EH^2)\} = 2\{(EF^2 + FG^2) + (GH^2 + EH^2)\} = 2\{(2\,EO^2 + 2\,OF^2) + (2\,EO^2 + 2\,OH^2)\}$ (Ex 245) $= 2\{4\,EO^2 + (2OF^2 + 2\,OH^2)\}$ (the diagonals of a ▱m bisect one another, $\therefore$ $EO = \frac{1}{2}EG$, and $HO = OF = \frac{1}{2}HF$), $\therefore$ $AC^2 + BD^2 = 2\{4(\frac{1}{2}\,EG)^2 + 4\,OH^2\} = \{2\,EG^2 + 4(\frac{1}{2}\,HF)^2\} = 2\{EG^2+HF^2\}$ (Ex. 233)

261. If an $\angle$ of a Δ be $=\frac{2}{3}$ of a rt $\angle$, then, the square on the side opposite to it, is = to the sum of the squares on the sides containing it, diminished by the rectangle contained by them *(Cal Ex Pap 1874)*

Let ABD be the Δ, in which $\angle ABD = \frac{2}{3}$ of a rt $\angle$ Then it is required to prove that $AD^2 = AB^2 + BD^2 - AB\,BD$ Draw AE $\perp$ BD From ED or ED produced, cut off EC=BE Join AC $\because$ BE=EC, AE com, and $\angle AEB = \angle AEC$, (Ax 11), $\therefore$ AB=AC, $\angle ABE = \angle ACE$ (I 4), $\therefore$ $\angle ACE = \frac{2}{3}$ of a rt $\angle$, the rem $\angle$ BAC of Δ ABC $= \frac{2}{3}$ of a rt $\angle$ (I 32), $\therefore$ $\angle BAC = \angle ACB$ (I 6), hence AB = BC = 2 BE Now $AD^2 = AB^2 + BD^2 - (2\,BE)\,BD$ (II 13) $= AB^2 + BD^2 - AB\,BD$

262 In a rt $\angle$ d Δ, the sum of the squares on the st lines drawn from the rt $\angle$ to the pts of *trisection of the hypotenuse* is = to five times the square on the st line between the pts of trisection

Let ABC be a rt $\angle$ d Δ, rt $\angle$ d at C The hypotenuse AB is trisected at E and D Join CE and CD To prove $CD^2+CE^2 = 5\,DE^2$ (*a*) In Δ CEA, $CA^2 + CE^2 = 2\,AD^2 + 2\,CD^2$ (Ex. 245) (*b*) In Δ CBD, $CD^2 + CB^2 = 2\,DE^2 + 2\,CE^2$ (Ex 245) Adding (*a*) and (*b*), we get, $(CA^2+CB^2) + (CD^2 + CE^2) = (2\,AD^2 + 2\,DE^2) + 2(CD^2+CE^2)$ (*c*), and $\because$ $\angle$ ACB is a rt $\angle$, $\therefore$ CA^2

$+CB^2=AB^2$, and ∵ $AD=DE$ (Hyp) ∴ $2AD^2+2DE^2=4DE^2$, ∴ $AB^2 = 4DE^2 + (CD^2 + CE^2)$ But $AB^2 = 9(\frac{1}{3}AB)^2=9DE^2$, $9DE^2=4DE^2+(CD^2+CE^2)$, ∴ $5DE^2=CD^2+CE^2$

263 **If from any pt within a rect st lines are drawn to the angular pts, the sum of the squares on one pair of the st lines drawn to opposite ∠s, is = the sum of the squares on the other pair**

Let ABCD be a rect, let the diagonals AC and BD intersect at P Now the diagonals of a rect are equal and bisect each other, ∵ $AP=BP=CP=DP$, and O is any pt within the fig, OA, OB, OC, OD are joined To prove $AO^2+OC^2=OB^2+OD^2$ Join OP (*a*) In △ AOC $AO^2+OC^2=2AP^2+2OP^2$ (Ex 245) (*b*) In △ OBD, $OB^2 + OD^2 = 2DP^2 + 2OP^2$ (Ex 245) Now $AP=DP$, $2AP^2=2DP^2$, ∴ $2AP^2 + 2OP^2 = 2DP^2+2OP^2$, $AO^2+OC^2=OB^2+OD^2$ from (*a*) and (*b*)

264 ***Three*** **times the sum of the squares on the sides of a △, is =** ***four*** **times the sum of the squares on the** ***medians***

Let ABC be a △, D, E, F, the mid pts of BC, AC, AB To prove $3(AB^2+AC^2 + BC^2)=4(AD^2+CF^2+BE^2)$ (1) $AB^2+AC^2 = 2AD^2 + 2BD^2$ (Ex 245), (2) $AC^2 + BC^2 = 2CF^2 + 2BF^2$ (Ex 245); (3) $CB^2+AB^2=2BE^2+2AE^2$ (Ex 245), adding (1), (2) and (3) $2(AB^2+BC^2+AC^2)=2(AD^2+CF^2 + BE^2) + 2(BD^2 BF^2+AE^2)$ or *doubles* of equals, are equal $4(AB^2+BC^2+AC^2) =4(AD^2 + CF^2 + BE^2)+4(BD^2+BF^2+AE^2)$, but $4(BD^2+BF^2 +AE^2) = BC^2+AB^2+AC^2$ (Ex 233), $4(AB^2 + BC^2 + AC^2) =4(AD^2+CF^2+BE^2)+(AB^2+BC^2+AC^2)$, ∴ $3(AB^2+BC^2+AC^2) =4(AD^2+CF^2+BE^2)$

265 *In the fig of II 11, show that* **four** *other st lines besides the given st line AB, are divided in* **medial section** **(Cam Ex Pap 1849)**

(See fig II 11, Text-book, p 138)

(1) From the figure it is evident, that rect CD DK$=CK^2$, for AB BH$=AH^2$ and $CD=AB$, $DK=BH$ and $CK=AH$

(2) CF FA$=AC^2$, for $AC=AB$

(3) KG GH$=HK^2$, for $KG=CF$, $KH=AC=AB$, and $GH=AF$

(4) If O be the pt of intersection of BE and HK, it is evident that △s BHO and ABE are *equiangular* and hence *similar*, and it can be proved by (Euc Bk VI 10), that AB and BE are *similarly divided* in H and O

266. **If ABC be a △ in which C is a rt ∠, and DE be drawn from a pt D in AC at rt ∠s to AB, prove that rect AB AE=rect. AC AD.** *(Cam Ex Pap 1861)*

$BD^2 = BA^2 + AD^2 - 2BA.AE$, $BD^2 = BA^2 + AD^2 - 2AD.AC$ (II. 13); $\therefore$ $2BA.AE = 2AD.AC$; $\therefore$ $BA.AE = AD.AC$.

267. *In II. 11, show that the pt. of section H, lies between the extremities of the st. line AB* (Cam. Ex. Pap. 1869).

(In the diagram of II. 11, Text.) $EF = EB$, and $\because$ $EF < EA + AB$; $\therefore$ $AF < AB$, $\therefore$ $AH < AB$, $\therefore$ H lies between A and B.

* 268. *In any trapezium, if two opposite sides be bisected, the sum of the squares on the two other sides, together with the squares on the diagonals is = to the sum of the squares on the bisected sides together with* four *times the squares on the st. line joining those pts. of bisection.*

Let AB, DC, two opp. sides of the trapezium ABCD, be bisected in E and F. Join EF, and draw the diagonals AC, BD. To prove $AD^2 + BC^2 + AC^2 + BD^2 = AB^2 + DC^2 + 4EF^2$. Join AF, BF. Since AF bisects DC the base of $\triangle ADC$, $\therefore$ $AD^2 + AC^2 = 2DF^2 + 2FA^2$ (Ex. 245); also $BC^2 + BD^2 = 2DF^2 + 2FB^2$ (Ex. 245); adding, $AD^2 + BC^2 + AC^2 + BD^2 = 4DF^2 + 2FA^2 + 2FB^2 = DC^2 + 2FA^2 + 2FB^2 = DC^2 + 2(FA^2 + FB^2) = DC^2 + 4AE^2 + 4EF^2 = DC^2 + AB^2 + 4EF^2$.

* 269. *From a given point in the* ⌒*ce of a semi-*⌒, *to draw a st. line meeting the diameter, so that the difference between the squares on this st. line and* $\perp$ *to the diameter from the point of intersection, may be = a given rectangle.*

Let A be the given pt. in the ○ce of the semi-⌒; from it draw AD $\perp$ to the diameter. Take O the centre, and divide DO in B, so that rect. $2OB.BD$ may be = the given rectangle. Join AB, and draw BC $\perp$ BD; AB, BC are the st. lines required. For $OA^2 = AB^2 + BO^2 + 2OB.BD$ (II. 12); $\therefore$ $AB^2 + 2OB.BD = OA^2 - OB^2 = OC^2 - OB^2 = BC^2$, $\therefore$ $BC^2 - AB^2 = 2OB.BD$, *i.e.* = the given rectangle.

270. *To describe a rectangular* ▭*m, which shall be = to a given square, and have its adjacent sides together = to a given st. line.*

Let AB = the given st. line. Upon it describe a *semi*-○ ADB. From A draw AC $\perp$ to AB, and = a side of the given square. Through C draw CD ∥ to AB, and let fall the $\perp$ DE. Then AE.EB will be the rectangle reqd. For $AE.EB = ED^2$ (II. 14) $= AC^2$ = the given square, and AB is the sum of the adjacent sides AE, EB.

271. If one $\angle$ of a $\triangle$, is = ⅔ of 2 rt. $\angle$s, show that the square on the opp. side, is > the squares on the sides forming that $\angle$, by the rectangle contained by the sides

Let ABC be a Δ, $\angle$ C = ⅔rd of 2 rt $\angle$s, it is required to prove that, $AB^2 = AC^2 + CB^2 + AC \cdot CB$. Draw CD $\perp$ BC produced. Since $\angle ACD$ = ⅔ of 2 rt $\angle$s, $\therefore \angle ACD$ = ⅓ of 2 rt $\angle$s, being supplementary $\angle$, CD = ½ AC (See Ex. 261). Now, $AB^2 = AC^2 + CB^2 + 2\,CB \cdot CD$ (II. 12). But ($2\,CB \cdot CD = 2\,CB$) (½ AC) $= CB \cdot AC$, $\therefore AB^2 = AC^2 + CB^2 + AC \cdot CB$.

272. In a Δ ABC, the $\angle$s B and C are acute; if BE, CE be $\perp$s to AC, AB respectively, show that $BC^2 = AB \cdot BF + AC \cdot CE$.

$AB^2 + 2AC \cdot CE = AC^2 + BC^2$ (II. 13), and $AC^2 + 2AB \cdot BF = AB^2 + BC^2$ (II. 13), by addition, $(AB^2 + AC^2) + 2\,(AC \cdot CE + AB \cdot BF) = (AC^2 + AB^2) + 2BC^2$. Hence $2\,(AC \cdot CE + AB \cdot BF) = 2BC^2$, $AC \cdot CE + AB \cdot BF = BC^2$.

273. Prove II. 10, by II. 7, II. 6, and Ex. 233.

Let AB be a st. line divided equally at P and produced to Q. Now $AQ^2 + BQ^2 = 2AQ \cdot QB + AB^2$ (II. 7) $= 2AQ \cdot QB + 4\,AP^2$ (Ex. 233) $= (2AQ \cdot QB + 2PB^2) + 2AP^2 = 2\,(AQ \cdot QB + PB^2) + 2AP^2 = 2PQ^2 + 2AP^2$, for $AQ \cdot QB + PB^2 = PQ^2$ (II. 6).

274. ABC is an isosceles Δ, in which AB = AC; AB is produced beyond the base to D, so that BD = AB; show that $CD^2 = AB^2 + 2\,BC^2$.

B is the mid. pt. of the base AD of Δ ACD, $\therefore AC^2 + CD^2 = 2\,AB^2 + 2\,BC^2$ (Ex. 245) $= AB^2 + 2AB^2 + BC^2 = AC^2 + AB^2 + 2BC^2$, for AC = AB, $\therefore CD^2 = AB^2 + 2\,BC^2$.

275. If MN be drawn ||l to the base BC of an isosceles Δ ABC, prove that $BN^2 - CN^2 = BC \cdot MN$.

Draw ND ||l AB, join BN, from N draw NX $\perp$ to CD. It is evident, that the fig. MBDN is a ▭m, hence MN = BD. Also it is evident, that NDC is an isosceles Δ. Hence X is the mid. pt. of CD, *i.e.*, CX = DX. Now $BN^2 = BX^2 + XN^2$ and $CN^2 = CX^2 + XN^2$ (I. 45), $\therefore BN^2 - CN^2 = (BX^2 + XN^2) - (CX^2 + XN^2) = BX^2 - CX^2 = (BX + CX)\,(BX - CX)$, (II. 5 Cor.) $= BC\,(BX - DX) = BC \cdot BD = BC \cdot MN$, for CX = DX (proved above), and BD = MN (I. 34).

276. If the base of a Δ be given, both in magnitude and position, and the sum of the squares on the sides in magnitude, the *locus* of the vertex is a ◯. (*Cf. Text p. 148, Ex. 30*)

Let BC be the given base of a Δ ABC, and $AB^2 + AC^2$ = a given square. It is required to find the *locus* of the vertex A. Bisect BC in D, join AD. Now $AB^2 + AC^2 = 2BD^2 + 2AD^2$ (Ex. 245), but $AB^2 + AC^2$ is *given* (hyp.), $\therefore 2BD^2 + 2DA^2$ is *given*,

and $2\,BD^2$ is *given*, since BD is *half* of the given base BC; $\therefore$ $2DB^2$ is *given*; $\therefore$ DA is *given*, and the pt D is *given*. Hence the *locus* of A is $\bigcirc$, having D as centre, and DA as radius

277. ABC is a $\triangle$, and O the pt of intersection of its *medians*, shew that $AB^2+BC^2+CA^2=3(OA^2+OB^2+OC^2)$

Let D, E, F be the mid pts of BC, CA, AB respectively; $\therefore$ AD, BE and CF are the **medians** In $\triangle$ OBC, $OB^2+OC^2=2OD^2+2BD^2$ (Ex 245), and doubles of equal things are equal, $\therefore$ (1) $2OB^2+2OC^2=4\,OD^2+4\,BD^2=AO^2+BC^2$ $\because$ $OD=\frac{1}{2}AO$ (*Cor. Ex 4, p 105 Text*) and $BD=\frac{1}{2}BC$ (Ex 233) So (2), $2OA^2+2OB^2=AB^2+OC^2$ So (3), $2\,OC^2+2\,OA^2=AC^2+OB^2$ Adding (1), (2), and (3), we see, $3(OA^2+OB^2+OC^2)+(OA^2+OB^2+OC^2)=(AB^2+BC^2+AC^2)+(OA^2+OB^2+OC^2)$, $\therefore$ $AB^2+BC^2+AC^2=3(OA^2+OB^2+OC^2)$

278 In the fig of II 11, if BE and CH meet at O, shew that AO is $\perp$ to CH

$\because$ EF = EB, $\angle$ EFB = $\angle$ EBF Now $\angle$ECO is the complement of $\angle$ EFL or $\angle$ EFB in $\triangle$ FCL (Ex 249). $\angle$ FLC $=1$ rt $\angle$, and $\angle$ EOC $=\angle$ LOB (I 15) = the complement of $\angle$EBF, $\therefore$ $\angle$ECO = complement of ($\angle$ EFB or $\angle$EBF) = $\angle$EOC, $\therefore$ EO=EC (I 6) EO=EA in $\triangle$ AOC since EO is the median dividing AOC into two isosceles $\triangle$s, $\angle$ AOC=($\angle$ ACO + $\angle$CAO) = a rt $\angle$ (I 32), AO is $\perp$ CL

279. If AB be divided in **medial section** *at C, prove that $(AB+BC)^2=5AC^2$*

$(AB+BC)^2=4\,AB\cdot BC+AC^2$ (II 8), but $AB\cdot BC=AC^2$ (Hyp) Hence $(AB+BC)^2=5\,AC^2$

280 If a straight line be divided equally and unequally, the squares on the two unequal segments = 2ce the rectangle contained by these segment = 4 times the square on the st. line between the pts of section

Let AB be the st line, divided *equally* at P, and *unequally* at Q It is required to prove $AQ^2+BQ^2=2AQ\cdot QB+4\,PQ^2$ Now $AQ^2+BQ^2=2\,AP^2+2PQ^2$, (II 9) $=2\,(AQ\cdot QB+PQ^2)+2\,PQ^2$, $\{$ for $AQ\cdot QB+PQ^2=PB^2$ or AP^2 (II 5) $\}$, $\therefore$ $AQ^2+BQ^2=2\,AQ\cdot QB+4PQ^2$

281 Construct a rectangle = the difference of two given squares

Let AC = a side of the *smaller* square, produce AC to D making CD = a side of the *greater* square and from CD cut off CB = CA. Then $AD\cdot DB+CB^2=CD^2$ (II 6), $\therefore$ $AD\cdot AB=$

$\cdot CD^2 - CB^2 = CD^2 - AC^2$ = difference of the two given squares. Thus the required rectangle is found

282. Find the side of a square = to a given equilateral △

Let ABC be the equil △ Draw BD ⊥ to AC, and BE, CE ∥ AC, BD respectively Then rect DE and △ ABC being each double of △ BDC, are equal Then find out the required side CG as in (II 14).

* *283 If on the radius AO of a ◯, a semi-◯ be described, and a ⊥ CDE to the common diameter ADO be drawn, (the square on the chord of the greater ◯ between the extremity of the diameter and the point of section of the ⊥) i e AE^2 will be = 2ce the sq on the corresponding chord of the lesser ◯ i e* $= 2AD^2$ (Cf. Cal Ex. Pap 1860, Q 5 *or* Ex 3)

Let ADO be the *semi*-◯ described on AO, from any pt C in AO or AB draw CDE at rt ∠s to AB cutting the *smaller* ◯ in D and the *larger* ◯ in E Join AE, BE, EO, AD, DO Bisect AO at X, join DX Now ∠ BOE = 2 ∠ OEA, and ∠ AOE = 2 ∠ OEB (I 32), ∠ BOE + ∠ AOE = 2 ∠s (OEA + OEB) = 2 ∠ AEB, but ∠ BOE + ∠ AOE = 2rt ∠s (I 13), 2 ∠ AEB = 2 rt ∠, ∴ ∠ AEB = a rt ∠, so ∠ ADO can be proved = a rt ∠, △ AEB and ∠ ADO are rt ∠d △s From △AEB, we have $AE^2 = AB\ AC$ (Ex 237), but AB = 2AO, $AE^2 = 2AO\ AC$, and from △ ADO, we have $AD^2 = AO\ AC$ (Ex 237), ∴ $AE^2 = 2\ (AO\ AC) = 2AD^2$

N B — *In the Q 5 or Ex 3 of 1860 — N, P, Q and C correspond to C, D, E and O here*

284. Construct a rectangle = to a given square, when the sum of the two adjacent sides of the rectangle, is = a given quantity

Let the st line AB be the *sum* of two adjacent sides of the rectangle, let CD represent the difference of two sides Then AB is *known* and it is reqd to find CD

$AB^2 \sim CD^2 = (A+B)\ (A \sim B)$, *i e* = 4 times the rect which is required, *i e* = 4 times the given square, *i e* = a known quantity. But AB^2 is known, thus CD^2 is known and ∴ CD is known. Hence, the sides of the required rect are found, for one side is $= \frac{1}{2}$ (AB + CD), and the other is $= \frac{1}{2}$ (AB ∼ CD)

**285 If from the extremities of any chord in a ◯, st lines be drawn to any pt in the diameter to which it is ∥, the sum of their squares = the sum of the squares upon the segments of the diameter*

Let AOB the diam, CD the chord, E the point in OB, and CF, DG ⊥s to AB, O the centre, join OC and OD, then $CE^2 = OC^2 +$

$OE^2+2OE \cdot OF$ (II. 12); $DE^2=OD^2+OE^2-2OE \cdot OG$ (II. 13), $= OC^2+OE^2-2 \, OE \cdot OF$, $\therefore$ $CE^2+DE^2=2OA^2+2OE^2=AE^2+EB^2$ (II 9).

286 *Construct a rectangle = a given square when the difference of two adjacent sides of the rectangle = a given quantity*

(Using the notation of Ex 284) we have $AB^2 \sim CD^2 = 4$ times the given square But CD is known, and $\therefore$ AB^2 is known, hence, the sides of the required rectangle are found

*287 *Produce one side of a given Δ, so that the rectangle contained by this side and the produced part, may be = the difference of the squares on the other two sides*

Let ABC be the given Δ, BC the side to be produced. Let CA be not > BA From A draw AD $\perp$ BC, or BC produced. Produce CD to E, so that DE may be = DC Then $CA^2 \sim BA^2 = CD^2 \sim BD^2$, (I. 47), rect (CD + BD) (CD ~ BD), (II 5 Cor) If the $\perp$ falls within the Δ ABC, CD + BD = BC, and CD ~ BD = BE, and if the $\perp$ falls without the Δ ABC, then CD + BD = BE, and (CD ~ BD) = BC. In each case, $CD^2 \sim BD^2 = BC \cdot BE$. Thus BE is the *produced part required*

288 *Find a square which shall be = to the sum of two given rectilineal figures.*

Describe the rects AB, CD = the given rectilineal figures (I 45). Apply the rect EF = CD, to the line BE (I 44), and describe a square = rect AF (II. 14), which shall be = the given rectilineal figures

289 *If the area of a rectangle be given, its* perimeter *is a* minimum, *when it is a square*

Let ABCD be a square and AEGH, a rect. of equal area. To prove that the perimeter of ABCD is < than that of AEGH. Now ABCD = AEGH (Hyp) Reject the common part AEFB, and we have EDCF = BFGH, hence these must be the complements about the diagonal of a ▱m, if DC, AF, HG be produced, they meet at the same point. Let them meet in K Now DK is > DA, $\therefore$ $\angle$DAK > $\angle$DKA *i e.* $\angle$CFK > $\angle$CKF, $\therefore$ CK > CF, and $\therefore$ CK > DE To each add CD + EA, and we have (KD + EA) or (GE + EA) > (CD + DA) Hence the *perimeter* of the rectangle is > than that of the square

290 *If two pts be taken in the diameter of a ⊙ equally distant from the centre, the sum of the squares on two st lines drawn from these pts to any point in the ⊙ ce, will be* constant

Let C, D be two points in the diameter AOB, then E being in the ⊙ ce, $CE^2+DE^2=2 \, OC^2+2 \, OE^2$ (Ex 245), which is *constant.*

291 *Given a Δ ABC, find the* locus *of the points, the difference of the squares of whose distances from B and C, (the ends of the base) is = to the difference of the squares on the sides AB, AC*

$\because$ Δ ABC is given, its sides AB, AC are given, $\therefore$ $AB^2 \sim AC^2$ is given. Find X, so that $X^2 = AB^2 \sim AC^2$ (I. 47).

*292. *Produce a given st. line, so that the sum of the squares on the given st. line and on the part produced, may be = 2ce the rect. contained by the whole st. line thus produced and the part produced.*

Let AB be the given straight line. From B draw BC at rt. $\angle$s to AB, making BC = AB. Join AC, and produce AB making AD = AC. Then $AB^2 + BD^2$ shall be = 2AD . BD. For $AD^2 + BD^2 = 2AD . DB + AB^2$ (II. 7), and $AD^2 = AC^2$ (const.) $= AB^2 + BC^2$ (I. 47) $= 2AB^2$ (constr.), hence $2AB^2 + BD^2 = 2 AD . DB + AB^2$, taking away AB^2 from these equals, we have $AB^2 = BD^2 = 2AD . DB$.

293. *Of the Δ ABC, the base BC is given, and the difference of the sides AB, AC, find the* locus *of the point where the $\perp$ from C to AC, meets the bisector of the* interior vertical *$\angle$ at A.*

Of all Δs whose base is the given base BC, and the differences of whose sides is given, let Δ ABC be one, and let CD be drawn $\perp$ AC meet the bisector of the interior vertical $\angle$ at D. From D draw DE $\perp$ AB or AB produced, and join BD. Now Δ ACD = Δ AED (I. 26), $\therefore$ AC = AE, and DC = DE, $\therefore$ BE = AB − AC or AC − AB, and is given. But $BE^2 = BD^2 - DE^2 = BD^2 - DC^2$, (I. 47. Cor.), $\therefore$ $BD^2 - DC^2$ is given, $\therefore$ the locus of D is a *straight line*.

294. *The* least *square which can be inscribed in a given square, is that which is* half *of the given square.*

Let ABCD be the given sq. and E, F, G, H the middle pts. of its sides. The fig. EFGH is a sq. Then sq. EFGH shall be = ½ABCD and < any other sq. KLMN inscribed in sq. ABCD. Join EG, FH, LN, then $\because$ EB, CG are = and ||l, $\therefore$ BC, EG are = and ||l (I. 33), and $\therefore$ Δ EFG = ½ ▭EBCG (I. 41). So Δ EHG = ½ ▭ EADG (I. 41). Hence the sq. EFGH = ½ sq. ABCD. And $\therefore$ OF is < OL (Ex. 58) and OH is < ON (Ex. 58), hence (OF + OH) or HF is < (OL + ON) or NL and $\therefore$ the sq. EFGH is < the sq. KLMN. So, it may be shewn that the sq. EFGH is < any other sq. inscribed in the given sq. ABCD. Hence the sq. EFGH is the *least*.

295. *Of the Δ ABC, the base BC is given, and the sum of the sides AB, AC, find the* locus *of the pt. where the $\perp$ from C to AC meets the bisector of the* exterior vertical *$\angle$ at A.*

Of all the Δs whose base is the given base BC and the *sum* of whose sides is given, let Δ ABC be one, and let CD drawn $\perp$ AC meet the bisector of the *exterior* vertical $\angle$ at D. From D draw DE $\perp$ BA produced, and join BD. Then Δ ACD = ΔAED (I. 26), $\therefore$ AC = AE, and DC = DE, $\therefore$ BE = BA + AC, and is given. Now $BE^2 = BD^2 - DE^2$ (I. 47. Cor.) $= BD^2 - DC^2$, $\therefore$ $BD^2 - DC^2$ is given, $\therefore$ the *locus of D is a straight line*.

*296 *If in the fig Euc I 47, the angular pts be joined, the sum of the squares on the* six *sides of the figure so formed, is=* eight *times the square on the hypotenuse*

Draw FN, KL, AM $\perp$s to DB produced, EC produced, and BC respectively; $\triangle$ AGH = $\triangle$ ABC, BC = GH In $\triangle$s BFN and ABM, (1) $\angle$ BFN + $\angle$ *FBN* = a rt $\angle$ (since they are *complementary* $\angle$s) = $\angle$ *FBN* + $\angle$ NBA; and since it is evident that DBN and AM are $\parallel$, $\therefore$ $\angle$ NBA = its *alternate* $\angle$ BAM (2); $\because$ from (1) and (2), $\angle$ BFN = $\angle$ BAM, and $\angle$ at N = a rt $\angle$ = $\angle$ at M, also BF = AB (being sides of the square FA), $\therefore$ $\triangle$ FBN = $\triangle$ ABM (I 26), $\therefore$ BN = BM. Again, in $\triangle$s KCL and ACM, KCL + $\angle$ *LKC* = a rt $\angle$ (since they are *complementary* $\angle$s) = $\angle$ *LKC* + $\angle$ ACM, $\therefore$ $\angle$ KCL = $\angle$ ACM, and $\angle$ at L = $\angle$ at M (being rt $\angle$s), also CA = CK, (being sides of the square AK), $\therefore$ $\triangle$ KCL = $\triangle$ ACM (I 26), $\therefore$ CL=CM.

In the obtuse $\angle$d $\triangle$ EBD, $FD^2 = DB^2 + BF^2 + 2$ DB BN (II 12)$= BC^2 + AB^2 + 2$ BC BM (3) In the obtuse $\angle$d $\triangle$ KCE, $KE^2 = EC^2 + CK^2 + 2$ EC *CL* (II 12) $= BC^2 + AC^2 + 2$ BC CM (4) Hence from (3) and (4), $FD^2 + KE^2 = 2\,BC^2 + (AB^2 + AC^2) + 2\,BC\,(BM + CM) = 2\,BC^2 + BC^2 + 2\,BC\,BC = 5\,BC^2$ (5), $(FD^2 + KE^2) + DE^2 + (HK^2 + GF^2) + GH^2 = 5\,BC^2 + BC^2 + (AC^2 + AB^2) + BC^2 = 5\,BC^2 + BC^2 + BC^2 + BC^2 = 8\,BC^2$

*297 ABCD is a quadrilateral, and X the middle pt of the st line joining the bisections of the diagonals, with X as centre any $\bigcirc$ is described, and P is any pt upon this $\bigcirc$, shew that $PA^2 + PB^2 + PC^2 + PD^2 = XA^2 + XB^2 + XC^2 + XD^2 + 4\,XP^2$

F, G are the mid pts of the diagonals AC and BD X and P are joined to the angular pts A, B, C, D In $\triangle$ APC, $PA^2 + PC^2 = 2\,AF^2 + 2\,PF^2$ (1), (Ex 245) In $\triangle$ PBD, $PB^2 + PD^2 = 2\,PG^2 + 2GB^2$ (2), (Ex 245) Adding (1) and (2), $(PA^2 + PC^2) + (PB^2 + PD^2) = (2\,AF^2 + 2\,PF^2) + (2\,PG^2 + BG^2) = (2\,AF^2 + 2\,BG^2) + 2\,(PF^2 + PG^2)$ In $\triangle$ PFG, $PF^2 + PG^2 = 2\,PX^2 + 2\,FX^2$ (Ex 245) or $2\,(PF^2 + PG^2) = 4\,PX^2 + 4\,FX^2$, *for doubles of equal things are equal*, $PA^2 + PB^2 + PC^2 + PD^2 = (2\,AF^2 + 2\,BG^2) + (4\,PX^2 + 4FX^2) = (2AF^2 + 2FX^2) + (2BG^2 + 2FX^2) + 4\,PX^2$ (3) In $\triangle$ AXC, $XA^2 + XC^2 = 2\,AF^2 + 2\,FX^2$ In $\triangle$ BXD, $XB^2 + XD^2 = 2\,BG^2 + 2\,GX^2 = 2\,BG^2 + 2\,FX^2$ (4) (Ex 245) from (3) and (4), $PA^2 + PB^2 + PC^2 + PD^2 = XA^2 + XB^2 + XC^2 + XD^2 + 4XP^2$

298 *Two rectangles have equal areas and equal* perimeters *Show that they are = in all respects*

Let the rectangles ABCD, EFGH have equal *perimeters* and equal *areas* The rect. ABCD shall be=rect EFGH, in all res-

pects. Produce BA until AK = AD, and FE until EL = EH. Bisect BK at M and FL at N (I. 10): KM=LN each being ½ of the perimeter of the rect. ∴ $KM^2 = LN^2$. But $KM^2 = BA.AK + AM^2$ (II. 5), and $LN^2 = FE.EL + EN^2$ (II. 5); ∴ $BA.AK + AM^2 = FE.EL + EN^2$. But $BA.AK = BA.AD = FE.EH = FE.EL$ (Hyp.); ∴ $AM^2 = EN^2$, and AM=EN. And ∴ AM = EN and BM = FN; hence AB = EF. But BK = FL; hence AK or AD = EL or EH; ∴ the rects. are equal in all respects.

299. *ABCD is a rectangle; P is a point such that PA + PC = PB + PD; show that the locus of P consists of two st. lines through the centre of the rectangle || to its sides.*

Let O be the *centre* of the rect. Since PA + PC = PB + PD, ∴ $PA^2 + PC^2 + 2 PA.PC = PB^2 + PD^2 + 2 PB.PD$ (II. 4). But $PA^2 + PC^2 = 2 PO^2 + 2 AO^2$ (Ex. 255); $PB^2 + PD^2 = 2 PO^2 + 2 BO^2$ (Ex. 255); so that $PA^2 + PC^2 = PB^2 + PD^2$. Hence PA.PC=PB.PD. Thus these two rects. have equal *areas* and equal *perimeters*, and ∴ (Ex. 298) must be = in all respects. Hence PA must be = either to PB or PD. If we take PA=PB, the pt. P falls on the st. line which is ⊥ to BC and passes through the *centre* of the rectangle. If we make PA=PD, the point P falls on the st. line which is ⊥ to AB, and passes through the *centre* of the rectangle.

300. *Given the difference of the squares of two st. lines, and the rectangle under them; to find the st. lines.*

Let a line DC be found (II. 14) whose square = the given diff. of sqs. and on it let a rect. CE be constructed = given rect. (I. 45); produce CD to A, so that CA.AD shall be = DE^2 (Ex. 255). From A *erect* AB=DE on the ⊥ DB, and draw BC; the required lines will then be BD and BC. For ∵ $AB^2 = CA.AD$; the ∠ABC is a rt. ∠; ∴ AB.DC=BD.BC. But AB=DE ∴ the rect. CE=BD.BC; and CE=the given rect.; $BD^2 - BC^2 = DC^2$, which is=the given difference of squares.

301. *Find a pt. in a given st. line, such that the sum of the squares on the lines from two given pts. to that point, shall be the least possible.* (Cal. Ex. Pap. 1891)

Let AB be the given st. line, and C and D the given pts. Join CD. Bisect CD in E (I. 10). From E drop EF ⊥ to AB meeting AB at F. Then F is the required point. Join CF and DF. Take any other point G in AB. Join CG, EG and DG. Now EG>EF (Ex. 58); ∴ $EG^2 > EF^2$; ∴ $CE^2 + EG^2 > CE^2 + EF^2$, ∴ $2 CE^2 + 2 EG^2 > 2 CE^2 + 2 EF^2$; but $2 CE^2 + 2 EG^2 = CG^2 + DG^2$ (Ex. 255), and $2 CE^2 + 2 EF^2 = CF^2 + DF^2$ (Ex. 255), ∴ $CG^2 + DG^2 > CF^2 + DF^2$. Similarly of any other pt. in AB. Hence $CF^2 + DF^2$ is the least.

BOOK III.

302 *If three pts A, B and C are not in the same st line, a ○ may be described whose ○ce shall pass through them*

Join AB and BC, bisect AB by the ⊥ DE (I 10 and 11), and BC by the ⊥ FE (I 10 and 11) The lines DE and FE will meet, for join DF, then (∠FDE + ∠DFE) < (∠BDE + ∠BFE) or 2rt ∠s (constr), and ∴ they will meet in some point E (Ax 12), then every ○ which passes through the points A and B, has its centre in ⊥ DE (III 1 Cor), and every ○ which passes through the points B and C, has its centre in the ⊥ FE (III 1 Cor), ∴ the ○ce of the ○ whose centre is E, passes through the *three* pts A, B and C

303 *Describe a ○ with a given centre cutting a given ○ at the extremities of a diameter*

Let ABD be a given ○, and O the centre of the ○ to be described Find C the centre of the given ○ (III 1), and join OC Draw the diameter AB at rt ∠s to OC, and join OA, OB The ○ described from the centre O and at the distance AO, will cut the given ○ at the extremities of the diameter AB For, in the △s ACO and BCO, AC = BC, CO is common, and ∠s at C are rt ∠s, hence AO = BO (I 4), and the ○ described from the centre O, and at the distance of any of them, will pass through the extremity of the other

304 *Show that the st line drawn at rt ∠s to the side of a quadrilateral inscribed in a ○, from their middle points, intersect at a fixed point*

Let EO, FO, GO, HO bisect ⊥ly the sides of the quadrilateral ABCD, inscribed in the ○ ABC Then, O the intersection of these lines, shall be a *fixed* point For EO bisects AB ⊥ly, the centre of ○ lies in this line (III 1 Cor) So, the centre of the ○ lies in each of the lines EO, FO, GO, HO (III 1 Cor) Hence, the pt. where they intersect, is the centre of the ○, and ∴ a *fixed* point.

305 *If a st. line drawn from the centre of a ○, bisect or be ⊥ to a chord, it will bisect and be ⊥ to all chords that are ||| to the former*

Let the st line CG, drawn from the centre C of the ○ ADEB, bisect the chord AB or be ⊥ to it, then it shall bisect and be ⊥ to any other chord, as D, which is ||| to AB For, if CF

bisects AB, it cuts it also at rt ∠s (III 3), hence, ∠AFC is a rt ∠, and ∠AFC=the interior ∠ CGD (I 29), hence CG is ⊥ to DE, and it also bisects DE (III 3) Again if CF be ⊥ to AB, it can be proved as before, that it is also ⊥ to DE, and consequently, it also bisects DE

306. If two ◯s cut each other, the st line joining the points of intersection, is bisected ⊥ly by the st line joining their centres

Let LMN, LMR, intersect in L and M, and let P and Q be their centres, then PQ shall bisect LM ⊥ly in O For draw the radii PL, PM, QL, and QM, then in the two Δs PLQ, PMQ, PL=PM, LQ=MQ, and PQ is common, and by (I 8) the Δs are equal in every respect, and ∴ ∠LPQ=∠MPQ. In the two Δs PLO, PMO, LP, PO are respectively=MP, PO, and ∠LPO=∠MPO, the Δs are equal in every respect (I 4), LO=OM, and ∠POL=∠POM, and they are rt ∠s (Def 7), hence LM is bisected ⊥ly by PO

307 If two ◯s cut one another, and from one of the points of intersection, a diameter be drawn through each of the ◯s, the other point of intersection and the other two extremities of the diameters, will be in one st line

Let LMN, LMR (See fig to Ex 306) be two ◯s, intersecting in L and M, and let LN, LR, be two diameters, then N, M, and R, shall be in one st line For join NM and MR, then ∠s LMN and LMR are ∠s in semi ◯, they are rt ∠s (III 31), and their sum=2 rt ∠s, . the st lines NM, MR, are in one st line (I 14)

308 A line that bisects two ||| chords in a ◯, is also ⊥ to them.

Let the line FG bisect the ||| chords AB, DE, then it shall be ⊥ to them For, if CF is a line joining the centre of the ◯ and F the mid pt of AB, it is ⊥ to AB, and if CF be produced, it will bisect DE ⊥ly in G (Ex 305), hence this line must coincide with the line FG, joining the middle points F and G, and hence FC is ⊥ to the two chords

309 If from any pt in the diameter of a ◯, st lines are drawn to the extremities of a ||| chord, the squares on these st lines are together=the squares on the segments, into which the diameter is divided

Let CD be drawn ||| to AB, the diameter of the ◯ ABC, and from any point E in AB, let EC, ED be drawn Then EC^2+ED^2 shall be $=AE^2+EB^2$ From O the centre of a ◯, draw OM ⊥ to CD; and ⊥ to AB. Join OC, EM, then CD is

bisected at M (III 3), $CE^2+ED^2=2CM^2+2ME^2$, (Ex 245) $=2CM^2+(2OM^2+2OE^2)$, (I 47) $=2(CM^2+OM^2)+2OE^2=2OC^2+2OE^2$ (I 47) $=2AO^2+2OE^2=AE^2+EB^2$ (II 9)

For the *other* proof, see Ex 285.

***310 A and B are two fixed pts without a ○ PQR, it is required to find a point P in the ○ce, so that the sum of the squares described on AP and BP, may be the *least* possible.**

Join AB, and from its mid pt D, draw DC passing through the centre of the ○, and cutting its ○ce at P Take Q, any other point on the ○ce, and join AQ, BQ then AP^2+BP^2 shall be $< AQ^2+BQ^2$ Join DQ, then $2AD^2+2DP^2=AP^2+BP^2$ (Ex 245), and $2AD^2+2DQ^2=AQ^2+BQ^2$ (Ex 245) But $2AD^2+2DP^2 < 2AD^2+2DQ^2$, for DP < DQ (III 8), hence AP^2+BP^2 < than the squares on the lines joining any other point to A, B Hence AP^2+BP^2 is the *least*

311 Two concentric ○s intercept between them, two equal portions of a st line, cutting them both.

Let PQN, RTS be two concentric ○s, and MN a line cutting them both, then the intercepted portions MR, NS, shall be equal For, from the *common centre* O, draw OV ⊥ to MN, then MN and RS are bisected in V (III 3), hence MV=VN and RV=VS, and ∴ MV−RV=VN−VS, *i e* MR=NS

312 If a chord to the greater of two concentric ○s, be a tangent to the less,—it is bisected in the point of contact

Let MPN, RTS (fig. to the Ex 311) be two concentric ○s, and PQ a chord to the greater ○ touching the less ○ in T, then PT shall be=TQ For, if OT be drawn, it will be ⊥ to PQ (III. 18), and since O is the centre of ○ MPN, and OT ⊥ to chord PQ, OT bisects PQ (III 3); ∴ PT=TQ

**313 Perpendiculars from the extremities of a diameter of a ○ upon any chord,—cut off equal segments*

Let ABDC be a ○, AB a diameter, CD a chord, and AE, BF ⊥s on the chord CD, then segment CE shall be=DF Draw GH through the centre O, ||| CD (I 31), and OI a ⊥ to EF. Then ∵ AE, OI, are ⊥s to EF, they are ||| (I 28), and EI, GO, are ||| by (constr); GI is a ▭m, and it is rectangular (I 46, Cor), So it may be shown that, IH is a rectangle; ∴ GO=EI, and OH=IF Now, in △s AOG, BOH, the vertical ∠s at O are equal (I 15), the alternate ∠s at A and B, are equal (I 29); and AO=OB, ∴ the △s are equal in every respect (I 26); hence GO=OH; hence EI=IF, and CI=ID (III 3), ∴ CE=DF (Ax 3).

314 **Through a given pt either *without* or *within* a given ◯, to draw a st line, the part of which intercepted by the ◯, shall be = to a given st line, not greater than the diameter of the ◯**

Let P be the given point *without* ◯ ABC, whose centre is O In the ◯, place a st line AB = given st line, bisect AB in E (I 10), and join OE With the centre O and radius OE, describe a ◯, this will touch AB in E, since ∠s at E are rt ∠s (III 3) from P draw PCD touching the ◯ in F (III 17); then PCD is the line reqd Join OF Now OF being = OE, CD will be = AB (III 14), *i e* = the given st line

315 **If any number of equal st lines be placed in a ◯, to determine the *locus* of their points of bisection**

Let there be any number of equal st lines AB, CD, placed in the ◯ whose centre is O, and let them be bisected in E and F (I 10), join OE, OF, then OE = OF (III 14), and the *locus* will be a ◯ whose centre is O, and radius = the distance of the pts of bisection from O

316 **If from a pt without a ◯, two st lines be drawn to the *concave* part of the ◯ce, making equal ∠s with the line, joining the same pt and the centre, the parts of the lines, which are intercepted within the ◯, are equal**

From the pt P *without* the ◯ ABC, let two lines PB, PD be drawn, making equal ∠s with PO, the line joining P and the centre then AB shall be = CD Let fall the ⊥s OE, OF, from O on PB and PD, then since ∠ at E = ∠ at F = a rt ∠, and ∠EPO = ∠ FPO, and the side PO, opposite to one of the equal ∠s in each, is common, OE = OF (I 26), and AB = CD (III 14)

317** **To draw a st line which shall touch two given ◯s —(1) If the ◯s be *equal

Let A and B be the centres, join AB, and from A and B, draw AC, BD at rt ∠s to AB, join CD, then AC being ∥ and = DB (I 28 and hyp) CD is ∥ to AB (I 33), ∴ CABD is a rectangular ▭, and ∠s at C and D being rt ∠s, CD is a tangent to both the ◯s (III 16)

(2) If the ◯s be unequal, and the line be reqd to touch them—on the *same* side of the line joining the centres

Let A and B be the centres, join AB, and with the centre B and distance = the difference of the given radii, describe a ◯, and from A, draw AE touching it (III 17) Join BE, and produce it to D through A, draw AC ∥ to BD (I 31), and join CD Then AC

being ||l and=DE, CD is=and ||l to AE (I 33); ∴ ACDE is a ▭, and ∠AEB being a rt ∠ (III 18), ∠AED is a rt. ∠. Hence ∠ C and ∠ D are rt. ∠s, and ∴ CD touches both ○s (III 16)

(3) If the line reqd to touch them—on *opposite* sides of the line joining the centres

With the centre B and radius=the sum of the given radii describe a ○, to which, from A draw a tangent AE (III 17) Join BE, and let it cut the given ○ in D Through A draw AC ||l BE (I 31), join CD Then AC being=and ||l ED, ACDE is a ▭ (I 33), and the ∠AED being a rt ∠, the ∠s at C and D are rt ∠s, and ∴ CD touches both ○s (III 16)

318 Of all st lines which can be drawn from two given points to meet on the *convex* ○ce of a given ○—the sum of those two will be the *least*, which make equal ∠s with the tangent at the point of *concourse*.

Let A and B be two given pts, CE a tangent to the ○ at C, where the lines AC, BC make equal ∠s with it, and let lines AD, BD be drawn from A and B to any other point D on the *convex* ○ce, then AC+CB shall be<AD+BD Let AD meet the tangent in E. Join EB, then AC+CB<AE+EB (Ex 3, p 243 Text), but AE+EB<AD+DB (I 21), ∴ AC+CB <AD+DB And, the same may be proved of st lines drawn to every other point in the *convex* ○ce

319 If a ○ be described on the radius of another ○, any st line drawn from the pt where they meet to the *outer* ○ce, is bisected by the *interior* one

Let ADB be a ○, described on the radius AB of the ○ ACE Draw any st line AC meeting the ○ ABD in D, then AD shall be=DC Join DB Then ∠ADB being in a semi-○, is a rt ∠ (III 31), and ∴ BD, being drawn from the centre B of the ○ ACE, bisects AC (III 3)

320 If a st line be drawn to touch a ○ and be ||l to a chord, the point of contact, will be the mid pt of the arc, cut off by that chord

Let CD be drawn touching the ○ ABE in the point E, and ||l to the chord AB, then E shall be the mid pt of the arc AEB Join AE, EB, now ∠ BAE=alternate ∠CEA, (I 29) and ∴ =to ∠EBA in the alternate segment (III. 32), hence, AE=EB (III. 28), and arc AE=arc EB

321. (1) Parallel st lines placed in a ○, cut off equal arcs (2) The two st lines in a ○, which join the extremities of two ||l chords, are equal to one another

(1) If FG be ||| AB Join AF, ∠FAB = ∠AFG (I 29), ∴ arc BF = arc AG (III 26)

(2) From (1), arc BF = arc AG, chord BF = chord AG (III 29)

322 From two given pts on the same *side of a st line of given position, to draw two st lines, which shall contain a given ∠, and be terminated in that st line*

Let A and B be the given pts, and CD be the given st line Join AB, and on AB describe a segment of a ◯ containing an ∠ = given ∠ (III 33), and (if the problem be possible), meeting CD in P, then P shall be the point reqd Join PA, PB, ∠APB being in the segment = the given ∠

323 If two chords of a given ◯, intersect each other, the ∠ of their inclination, is > ½ the ∠ at the centre, which stands on an arc = the *sum or difference* of the arcs intercepted between them, according as they meet *within* or *without* the ◯

Let AB, CD each cut one other in the point E, (1) *within* the ◯ ABC, then ∠ of inclination shall be = ½ the ∠ at the centre, standing on an arc = CA + DB Through D, draw DF ||| to BA Find O the centre of the ◯ (III 1), join CO, FO Then AB being ||| to ED (Ex 320), AF = BD, and ∠ CEA = ∠CDF (I 29), = ½ ∠COF (III 20), which stands on the arc CF = CA + BD

(2) Next, let AB, CD intersect in E, *without* the ◯ The same construction being made, ∠ CEA = ∠ CDF (I 29) = ½ ∠ COF (III 20) = ½ the ∠ standing on CF, which is the difference between CA and AF, or CA and BD

**324 If the ◯ce of a semi-◯, be divided into an* odd *number of equal parts, and through the pts which are equally distant from the diameter, st lines be drawn, the segments of these st lines, intercepted between radii drawn to the extremities of the most remote, will together be = to a radius of the ◯*

Let the ◯ce of the semi-◯ ADB be divided into any odd number of equal parts, *e g five*, (the proof being the same for any *odd* number) in the points C, D, E, F Join DE, CF which are |||, join CE, DE, CF intercept equal arcs CD, EF, ∴ ∠DEC = ∠ECF (III 27), and these are alternate ∠s, DE is ||| CF Join OD, OE, then DE + LM shall be = the radius of the ◯ Complete the ◯, and divide the opposite semi-◯ in the same manner, join AC, DG, EH which will be ||| to one another, CH will also be ||| to DI. Hence DE = OK, and OK is also = each of

the two, PM, CL (I 34), ∴ PM=CL, hence, taking away the common PL from both, LM=CP, which=AK, since CF is ∥ to AB, and DE+LM=KO+AK=AO, the radius of ○

325 If two equal ○s, cut each other, and from either pt of intersection, a ○ be described cutting them, the pt where this ○ cuts them and the other pt of intersection of the equal ○s, are in the same st line

Let the two equal ○s cut each other in A and B, and with the centre A and any distance AC, describe a ○ FCD cutting their ○ces in C and D, then C, D, B shall be in a st line Join CB, and let it meet the ○ce ADB in E Join AE, AC Since ∠ABC is an ∠ in each of the *two equal* ○s, arc AC=arc AE (III 26), ∴ AC=AE (III. 29), and ∴ E is the point in ○ FDC, and being (by const.) in ○ce ADB, it must coincide with D, ∴ CB passes through D, or C, D, B are in a st. line

326 If two equal ○s cut each other, and from either point of intersection, a st. line be drawn meeting the ○ces, the part of it intercepted between the ○ces, will be bisected by the ○, whose diameter is the common chord of the equal ○s.

Let the equal ○s ADB, ACB cut each other in A and B, join AB, and on AB as diameter, let a ○ AEB be described, and from A draw any line ADC meeting the ○ces in D and C, then DC shall be bisected in E Join BD, BE, BC Since ∠ CAB is in each of the two equal ○s, arc BD=arc BC (III 26), and ∴ chord BD = chord BC and ∴ ∠BDE = ∠ BCE (I 5), and ∠ BED in a semi-○ is a rt ∠ (III 31) and ∴ = ∠BEC, and BE is common to the two Δs, BED, BEC, ∴ DE=EC (I 26)

327 If two ○s touch each other **externally or internally** *any st line drawn through the pt of contact, will cut off similar segments*

Let the ○s ADC, BCE touch each other in the point C, and let any st line ACB be drawn through the pt of contact, it will cut off similar segments. Let the ○s ADC, BCE touch each other in the point C, and let any line ACB be drawn through the pt of contact, it shall cut off similar segments Draw the diameters CD, CE, and join AD, BE Then DCE being a st line (III 12)—∠ ACD=∠BCE (I. 15), and ∠DAC= ∠CBE=a rt. ∠, each being in a semi-○, (III 31), hence ∠ ADC=∠ CEB (I 32, Cor), and the segments ADC, CEB are similar, and ∴ the segments AC and CB are also similar

328 If two circles touch each other **externally or internally,** *two st lines drawn through the pt of contact, will intercept arcs, the chords of which are* ∥.

Let two ◯s ACD, ECB touch each other in C, and let ABC, DEC be any two st lines drawn through the point of contact Draw the tangent FCG (III 17), and join AE, EB Now ∠ADC = ∠FCA (III 32) = ∠ BEC, ∴ AD is ∥ to BE (I 28)

329. If from the pt of contact of two ◯s, which touch each other **internally**, *any number of st lines be drawn, and through the pt where these intersect the ◯ces st lines be drawn from any other pt in each ◯ce, and produced to meet, the ∠s formed by these st lines, will be equal*

Let the two ◯s ABC, DEC touch each other *internally* in C, from which, let any lines CA, CB be drawn, and taking any two pts G and F draw GEI, FBI through E and B, and through D and A, draw GDH, FAH, if those st lines meet, the ∠ at I shall be = ∠ at H For ∠ CBF = ∠ CAF (III 21), standing on the same arc CF, ∴ ∠ IBE = ∠HAD, *since they are supplementary to equal ∠s* Also ∠ CEG = ∠ CDG (III 21), standing on the same arc CG, and ∠ IEB = ∠ HDA (I 15), ∴ Δs IEB, HDA have two ∠s in each equal, and the remaining ∠s are equal, *i e* ∠ EIB = ∠ DHA (I 32, Cor)

330 In the diameter produced of a ◯—to determine a point, from which a tangent drawn to the ◯ce shall be = to the diameter

From A the extremity of the diameter AB, draw AD at rt ∠s and = to AB (I 11, 2) Find O the centre (III 1), join OD cutting the ◯ in C, and from C draw CE at rt ∠s to OD (I 11) meeting BA produced in E Then ∵ ∠ OAD = ∠ OCE, each being a rt ∠ (const), and the ∠ at O, is common to two Δs OAD, OCE, and OA = OC = a radius, ∴ AD = CE (I 26) But AD was made = AB, ∴ CE = AB and CE is a tangent at C (III 16), ∴ E is the point required

331 Given two equal chords AB, AC of a ◯ BAC shew that AD, which bisects the ∠BAC, passes through the centre

Join BC Let AD meet BC at E In Δs ABE, ACE, ∵ AB = AC, ∠ABE = ∠ ACE, ∠ BAE = ∠ CAE (Hyp), ∴ BE = EC (I 26), and ∠ AEB = ∠ AEC, but these are adjacent ∠s, ∴ they are rt ∠s, ∵ AD bisects BC, a chord, at rt ∠s, ∴ it must pass through the centre (III 3)

332 Find the locus of the centres of ◯s, which pass through two given pts

Let A and B be two given points Join AB, and bisect it at C Through C, draw CD at rt ∠s to AB The *unlimited* line CD will be the reqd *locus* On CD, take any point E, and join EA and EB Now ∵ AC = CB (const) CE is com and ∠ ACE

= ∠ BCE = a rt ∠. ∴ AE=BE (I. 4), ∴ if with centre E and distance=EA, a ◯ be described, it shall pass through B The same can be shewn for any other point on CD, or CD produced, ∴ the *unlimited* st line CD which bisects AB at rt ∠s, is the *locus* of centres of all ◯s passing through A and B

N B—Here, evidently AB is a chord, and ∵ CD bisects AB at rt ∠s, the centres of ◯s lie on CD (III 3)

333 Two chords AB, CD are given in position and magnitude, find the centre of the ◯ (See Notes on III 1)

334. Describe a ◯ that shall pass through two given points, and have a given radius

Let A, B be the two given points, and C the given radius It is required to describe a ◯, that shall pass through A and B, and have a radius=C With A as centre and radius=C, describe a ◯, and with B as centre and radius=C, describe another ◯, let these ◯s intersect one another at D, then evidently DA = C = DB, now, if with D as centre, and radius=C, a ◯ be described, then this ◯ is the reqd. ◯, for it passes through the given pts. A and B, and its radius (FA or FB)=C, the given radius

335 Through a given point within ◯, draw a chord, which shall be bisected at that point.

Find the centre E (III 1) Join EA. Through A draw AD at rt. ∠s to EA (I 11), and produce DA to B, to meet the ◯ce at B Then BD shall be the reqd chord, ∵ EA passing through the centre, cuts the chord BD at rt ∠s, it bisects the chord (III 3), ∴ BD is drawn, bisected at A

336 Find the locus of the middle points of a system of parallel chords drawn in a ⊙

Let AB be a chord in a ◯ ABC To find the *locus* of middle points of all chords parallel to AB Bisect AB at D (I 10) Find E the center (III 1) Join ED Produce DE to F and G to meet the ◯ Then FG shall be the reqd *locus* Draw CK any chord ||| to AB, cutting FG at H. Now ∠ KHD + ∠HDA=2 rt ∠s (I 29), ED passing through the centre, bisects AB, ∴ ∠ EDA or ∠ *HDA*=1 rt ∠ (III 3), ∴ ∠KHE or ∠KHD=1 rt ∠, but ∠EHK+∠EHC=2 rt ∠s, ∴ ∠EHC= 1 rt ∠, ∴ KC is bisected at H (III 3). So, it may be shewn that the middle point of any chord ||| to AB, lies on FG, ∴ FG is the required *locus*

337. Two ◯s which intersect at one point, must also intersect at another

Let ◯s ABC and DBC intersect at B Find the centres E and F, and join EF From B draw BH at rt ∠s to EF; and produce BH to meet ◯ ABC at C Join EB, EC, FB and FC Since BC is a chord of ◯ ABC, and EH is drawn through E the center of ◯ ABC at rt ∠s to BC, BC is bisected at H (III 3) Now, in Δs FHB, FHC, BH=CH (proved), FH is common, ∠BHF = ∠CHF = a rt ∠, BF = CF, but BF is a radius of ◯ DBC, FC is also radius of ◯ DBC, hence C, which is on the ◯ce of ◯ ABC, is also a pt of the ◯ DBC; ∴ the ◯s also intersect at C

338 If two ◯s, whose centres are A and B, touch one another (1) externally or (2) internally at C, and a st line be drawn through their point of contact, cutting the ◯ce at P and Q,—show that AP and BQ are |||

(1) Join AB, AB shall pass through C the pt of contact (III 12), ∠ APC = ∠ ACP, ∵ AP = AC (radii), ∠ BCQ = ∠ BQC, ∵ BC=BQ (radii), ∠ ACP = ∠BCQ (I 15), ∴ ∠APC or ∠ APQ = ∠ BQC or ∠ BQP and these are *alternate* ∠s, AP is ||| to BQ (I 28)

(2) Join AB, and produce it AB passes through C the pt of contract (III 11), ∠ APC = ∠ ACP, ∵ AC=AP (radii), ∠ BCQ = ∠ BQC, ∵ BC=BQ (radii) But ∠BCQ or ∠ ACP = ∠ APC, ∠ APC = ∠ BQC, and these are *corresponding* ∠s, AP is ||| to BQ (I 28)

339 Find the *locus* of the centres of all ◯s, which touch a given ◯ ABC, at a given point B

Find O the centre of ◯ ABC (III 1), join OB The *unlimited* line OB is the required *locus*, because it is the st line that joins the centre of any ◯ to that of ABC, for it passes through the point of contact (III 11 and 12)

340 Describe a ◯ to pass through two given points A, B and have its centre on a griven st line CD When is this impossible?

Join AB, and bisect it at E (I 10) Through E draw EF at rt ∠s to AB (I 11), meeting CD at F, then F shall be the centre of the required ◯ Join AF, BF, ∵ AE=BE, EF is common, and ∠ AEF = ∠BEF, ∴ ΔAEF = Δ BEF (I 4), ∴ if, with centre F, and distance AF a ◯ be described, it shall pass through B, ∵ AF=BF, being the bases of Δs AEF, BEF

N B—It is impossible to describe such a ⊙, when EF does not meet CD *i e.* EF is ∥ CD

* *341. Of the* five *conditions—(1) of passing through the centre of a* ⊙, *(2) bisecting a chord, (3) being* ⊥ *to a chord, (4) bisecting the subtended* ∠ *at the centre, (5) bisecting the subtended arc of the chord—if a st line fulfil any* two, *it will also fulfil the other* three

Let AMB be a ⊙, AB a chord, and CF a st. line bisecting AB, or cutting it at rt ∠s, or bisecting the arc AMB

1. If CF bisect AB, it cuts AB ⊥ly (III 3).

2 If CF cut AB ⊥ly, it bisects AB (III 3)

3 If CF bisect AB ⊥ly, it passes through the centre (III 3, Cor)

4 If CF bisect AB, or cut AB at rt ∠s, it will bisect ∠ACB, and also the arc AMB For the Δs ACF, BCF, are every way equal (III 3) in both these cases, and hence ∠ACM = ∠BCM , and ∴ the arc AM = arc MB (III 26)

5. If CM bisect ∠ACB, or the arc ACB, it will also bisect AB, and cut it at rt ∠s For, if ∠ ACF = ∠BCF, then since AC=CB, and CF common to the Δs ACF, BCF—the Δ ACF = Δ BCF (I 4) and ∴ AF=FB, and ∠AFC = ∠BFC, and they are rt. ∠s Also, when ∠ACB is bisected, the arc AM =arc MB, for they subtend equal ∠s at the centre (III 27) Again, if CM bisects the arc AMB, ∠ACM = ∠BCM, ∵ they stand on equal arcs, and ∴ it can be proved, as above, that CM also bisects AB, and is ∴ ⊥ to it

6 If the st. line FM bisect the arc AMB, and also AB, it will cut AB at rt ∠s , it will pass through the centre C, and will bisect ∠ACB For, it has been proved that a st line CM, drawn from the centre to M, (the mid. pt of the arc) bisects AB , hence the line MF drawn from M to F, must coincide with the former, and hence, will pass through the centre, will also cut AB at rt ∠s, and bisect ∠AGB

7 If the st line MF bisect the arc AMB, and cut AB at rt ∠s, it will bisect AB, will pass through the centre, and will bisect ∠ACB

This case can be proved exactly in the same manner as the preceding

8 If CM bisect the arc AMB, and also ∠ACB, it will bisect AB, and be ⊥ to AB, and will evidently pass through the centre C Since the st line must in this case, pass through C, and as it also bisects ∠ACB, this case is thus reduced to the *first part of the*

fifth case above The *fourth case* above properly contains *two cases* and so does the *fifth*, there are thus, in all, *ten cases*

342 If two chords in a ○ intersect each other ⊥ly, the sum of the squares on the four segments is = the square on the diameter

Let ABDC be a ○, and AB, CE two ⊥ chords in it, then the sum of the squares on the segments AF, FC, EF, FB is = the square on the diameter Draw the diameter ED, and join CD, AC, DB, and EB Then ∠ECD is a rt ∠ (III 31), and ∴ = ∠EFB, hence CD is ∥ to AB (I 29), and ∴ arc AC = arc DB, ∴ chord AC = chord DB (III 26) Also ∠EBD in a semi-○, is a rt ∠ (III 31) Now $AF^2+FC^2=AC^2$ (I. 47) $=DB^2$, and $EF^2+FB^2=EB^2$ (I 47), also $EB^2+DB^2=DE^2$ (I 47), ∴ adding equals to equals, $AF^2+FC^2+EF^2+FB^2=DE^2$

343. ○s are described on the sides of a quadrilateral as diameters, show that the common chord of any adjacent two, is ∥ to the com chord of the other two

Let ○s ABQ, BPC, CPD, DQA be described on the sides of a quadrilateral ABCD, as diameters Then AQ, CP shall be ∥ Join BP, PQ, DQ, then ∠BPC = ∠CPD = is a rt ∠ (III 31), ∴ P is in the st line BD It may be shown that, Q is in the st line BD (I 14) Hence AQ, CP are ∥ (I 27), for ∠s AQP, QPC are rt ∠s (III 31) So the com chords of the other pair of adjacent ∠s, may be shown to be ∥

**344 Through a given pt without a given ○, to draw a st line to cut the ○, so that the two ⊥s drawn from the points of intersection, to that diameter which passes through the given pt may together be = a given st line not > than the diameter of the ○*

Let P be the given point without the ○ ABC, whose centre is O, and AB the diameter which passes through P On PO describe a semi ○ From P, draw PD at rt ∠s to PB (I 11) and = ½ the given line through D draw DE ∥ to PB (I 31), meeting the semi ○ in E, join PE, and produce it to C then PC is the st line reqd For, draw FG, EH, CI ⊥s to AB (I 12) Join OE, then ∠PEO is a rt ∠, and ∠OEC = a rt (III 31), and ∴ EF = EC (III 3), hence FG + CI = 2EH (Ex 192) = 2PD (I 34) = the given st line

**345 If from each extremity of any number of equal adjacent arcs in the ○ce of a ○, st lines be drawn through two given points in the opposite ○ce, and produced till they meet, the ∠s formed by these st lines, will be equal*

Let AB, BC, be *equal* arcs, and F, E, two pts in the opposite ◯ce, through which let the lines AFI, BEI, BFH, CEH be drawn, so as to meet, then ∠s at I and H, will be equal From E draw EK, EL, respectively ∥l to FA, FB, (I 31) Since EK is ∥l to FA, ∠KEB = ∠ at I, and ∠ LEC = ∠ at H (I 29) But since arc AB = arc BC, and AK = BL, being each = EF (*Ex 320*), KB = LC and ∠ KEB = ∠LEC (III 27), ∴ also ∠s at I and H are equal The same may be proved, whatever be the number of equal arcs AB, BC

346 If from any pt without a ◯, st. lines be drawn touching it, the ∠ contained by the tangents, is twice the ∠ contained by the line joining the pts of contact and the diameter drawn through one of them.

From pt E without the ◯ ABC,—let EA, ECD be drawn touching the ⊙ in A and C (III 17), and let ED meet the diameter AB, drawn from A in the pt D Join AC, ∠AEC = 2 ∠CAB Through C draw the diameter COF, then ∠FCD is a rt ∠ (III 18), and ∴ = ∠EAD, and ED ∖ is common to the △ EDA, COD, ∴ ∠BOD = ∠AED But ∠COB is = 2 ∠CAD, ∴ ∠AEC is = 2 ∠CAD

347. If from the extremities of the diameter of a ◯, tangents be drawn and produced to intersect a tangent to any point of the ◯ce, the st lines joining the points of intersection and the centre of the ◯, form a rt ∠

From A and B the extremities of the diameter AB, let tangents AD, BE be drawn, meeting a tangent to any other point C of the ◯ce, in D and E, and let O be the centre Join DO, EO, then ∠ DOE shall be a rt ∠, join CO Now ∵ CE = EB, CO = OB (III 17, Cor), and ∠ at C = ∠ at B, (being rt ∠s), ∠CEO = ∠OEB and ∠CEB is bisected by EO So ∠ADC is bisected by DO And ∵ ∠CEB + ∠CDA = 2 rt ∠s, ∴ ∠CDO + ∠CEO = one rt ∠, ∴ ∠DOE is a rt ∠ (I 32)

**348 Two ◯s being given in magnitude and position, to find a point in the ◯ce of one of them, to which, if a tangent be drawn cutting the ◯ce of the other, the part of it intercepted between the two ◯ces, may be = a given st line*

Let O and C be the centres of the two given ◯s To any pt A in the ◯ce of one of them, let a tangent AB be drawn, and make AB = the given st line With the centre C and distance CB, describe a ◯ DBD cutting the other in the pt D, and from D draw DE touching the former given ◯ (III 17); then E shall be the point required Join CA, CB, CD, CE, ∴ CA = CE

and CB = CD, and ∠s at A and E are rt ∠s, ∴ DE = BA, *i e* = to the given st line

N B —If the ○ DBD neither *cuts* nor *touches* DD, the problem will be impossible

*349 *To draw a st line cutting two concentric* ○*s, so that the part of it which is intercepted by the* ○*ce of the* greater, *may be* double *the part intercepted by the* ○*ce of the* less

Let O be the centre of the two ○s Draw any radius OA of the *lesser* ○, and produce it to B, making AB = AO On AB describe a semi-○ ACB, cutting the greater ○ce in C Join AC, and produce it to E Then CE is the st line required Join CB, and let fall ⊥ OD Then ∠ADO being a rt ∠ (const) = ∠ACB (III 31), and the *vertically opp* ∠s at A are equal (I 15), and OA = AB, ∴ AC = AD and DC = 2AD, but DC = ½ EC, and AD = ½ AF, ∴ EC = 2AF

N B —The same construction will apply whatever be the relation required between the two chords

*350 *If a semi-*○ *be described on the side of a* quadrant, *and from any point in the* quadrantal arc, *a radius be drawn, the part of this radius intercepted between the* quadrant *and semi-*○, *is = the* ⊥ *let fall from the same point on their common tangent*

On AB the side of a *quadrant*, let the semi-○ AEB be described, and from any pt C, draw the radius CB, and CD ⊥ to AD (a tangent at A), then EC shall be = CD Join AE, AC, then ∠ AEB being in a semi-○, its adjacent ∠ AEC is a rt ∠ (III 31), and = ∠ ADC, and ∠ BCA = ∠ BAC = ∠ ACD, the *alternate* ∠ (I 29), ∴ two Δs AEC, ACD have two ∠s in each equal, and one side AC is common, ∴ EC = CD (I 26)

Cor Any chord of the semi-○, drawn from the centre of the quadrant = the ⊥ drawn to other side from the point in which the chord produced meets the *quadrantal arc* Produce DC to F, then CE being = CD, the remainder BE = remainder CF

*351 *If the chord of a quadrant, be made the diameter of a semi-*○, *and from its extremities two st lines be drawn to any pt in the* ○*ce of the semi-*○, *the segment of the* greater *line intercepted between the two* ○*ces, shall be = the* less *of the two st lines*

Let O be the centre of the *quadrant* ADB Join AB and on AB, let the semi-○ ACB be described, from any pt C let st lines CA, CB be drawn to A and B, of which CB is the *greater*, then CD shall be = CA Join AD and complete the

○ ABE, take any pt E, and join EA, EB. Since ADBE is a quadrilateral figure inscribed in a ○, ∴ ∠AEB + ∠ADB = two rt. ∠s (III. 22), and ∴ = ∠ADB + ∠ADC. hence ∠AEB = ∠ADC. But ∠AEB = ½ ∠AOB (III. 20), which is a rt ∠, ∴ ∠ADC = ½ a rt ∠, and ∠ACD or ∠ACB being a rt ∠ (III. 31), ∠CAD = ½ a rt ∠, and ∴ = ∠CDA; ∴ CA = CD (I. 5).

***352. If two ○s cut each other, so that the ○ce of one passes through the centre of the other, and from either point of intersection a st. line be drawn cutting both ○ces, the part intercepted between the two ○ces, will be = the chord drawn from the other point of intersection, to the point where it meets the inner ○ce.**

Through O the centre of the ○ ABC, let ○ AOB, be described cutting the ○ ABC in A and B. If any st. line AED be drawn from A, and BE joined, then DE will be = EB. Draw the diameter AOC, join BC, BD, ∴ ∠AOB = ∠AEB (III. 20), ∴ ∠COB = ∠DEB (∴ they are *supplementary* to equal ∠s). Also ∠OCB = ∠EDB (III. 21), ∴ △s OCB, EDB are *equiangular*, and ∴ OB = OC, ∠OCB = ∠OBC, hence ∠EDB = ∠EBD, and ∴ ED = EB (I. 6).

353. *If a pt be taken without a ○, and from it tangents be drawn to the ○, and a 2nd pt be taken in the ○ce between the two tangents, and a tangent be drawn to it, the sum of the sides of the △ thus formed, is = the sum of the two tangents.*

From a given pt D, let two tangents DA, DB be drawn; and to C any pt. in the ○ce between them, let a tangent ECF be drawn. Then the sum of the sides of the △, shall be = the sum of two tangents i. e. = DA + DB. Since AE = EC, and FC = FB (III. 17, Cor.), ∴ DE + EF + FD = AD + DB. ∴ if through any other pt in the arc ACB, a tangent be drawn, it will be = the two segments of DA, DB intercepted between it, and the pt of contact A and B, and the *three* sides of the △ formed, will be = DA + DB.

354. *Of all △s on the* same *base, and between the same parallels, the* isosceles *has the* greatest vertical ∠.

Let ABC be an isosceles △, on the base AC and between the ||s AC, BD. It has a *greater* vertical ∠ than any other △ADC on the same base and between the same ||s. Through A, B, C describe a ○ ABC (Ex. 302); ∴ chord AB = chord CB, being sides of an isosceles △, ∴ *arc* AB = *arc* CB (III. 28), and since B is the mid. pt. of the arc, and BD is ||| AC;—BD is a tangent at B. Let the arc cut AD in E; join EC. Then ∠ABC = ∠AEC (III. 21), and ∴ is greater than ∠ADC.

N B—Of all Δs on the same base and having the same vertical $\angle$, the *isosceles is the greatest* For the ΔAEC has the same vertical $\angle$ with ΔABC, and ΔABC=ΔADC, on the same base and between the same ||s (I 37), but ΔADC > ΔAEC, ΔABC > ΔAEC

355 Given one $\angle$, a side opposite to it, and the sum of the other two sides, to construct the Δ

Let AB be the given side Upon AB describe a segment of a ◡ containing an $\angle=\frac{1}{2}$ the given $\angle$ (III 33), and from A draw AC = given sum of the two sides, join BC, and make $\angle$CBD = $\angle$BCD, then ABD is the Δ reqd Since $\angle$DCB= $\angle$DBC, DB=DC, . AD + DB=given sum, and $\angle$ADB= $\angle$DBC + $\angle$DCB (I 32), *i e* = 2 $\angle$DCB, and = the given $\angle$

356 If two chords of a ◯ intersect at rt $\angle$s, the portion of the ◡ce taken alternately, are together = to $\frac{1}{2}$ of the ◯ce

(C U Pap 1895, II q 3)

Let the chords XY and NM intersect in R and at rt $\angle$s to each other Find Q the centre (III 1) Join MQ, and produce MQ to meet the ◡ce at P, join PN, XN and YN Then $\angle$PNM is a rt $\angle$ (III 31), PN || XY (I 28), hence $\angle$ PNX = $\angle$ NXY (I 29), the arc XP=the arc YN (III 26) Add to each of the equals, the arc PN + MY, then the arc XN + MY=the arc PNM, which is=$\frac{1}{2}$ of the whole ◯ ce

357 If two ◯s cut one another, find a pt from which the st lines drawn to touch the two ◯s shall be equal to one another

(C U. Pap 1859, II q 4)

Let two ◯s cut each other in P and Q The st lines drawn from any point in QP produced, to touch the two ◯s are = one another Let X be any pt in QP produced Draw XY and XZ tangents to the two ◯s PQY and PQZ (III 17), . YX^2=PX XQ (III 36), and ZX^2=PX XQ (III 36), YX^2= ZX^2, YX=ZX

358 If on the radius AO of a ◯ a semi-◯ be described, and from any point N in AO the diameter of the semi-◯, a ⊥ NPQ be drawn to meet the ◯s in P and Q, then if the common extremity A of their diameters be joined with these points, AQ^2=$2AP^2$

(C U Pap 1860, q 3)

Join QB, ∵ $\angle$ AQB is a rt. $\angle$ (III 31), ∴ AB^2=AQ^2+ BQ^2 (I 47)=AN^2 + NB^2 + $2NQ^2$, also AB^2 = AN^2 + NB^2+ 2AN.NB (II 4), 2 NQ^2=2AN NB or NQ^2=AN.NB Add to

each of these equals AN^2, then $AN^2 + NQ^2 = AN^2 + AN \cdot NB$ But $AN^2 + NQ^2 = AQ^2$ (I 47), and $AN^2 + AN \cdot NB = AB \cdot AN$ (II 3); $AQ^2 = AB \cdot AN = 2AO \cdot AN$

If PO be joined, it can be proved that. $AP^2 = AO \cdot AN$ Hence $AQ^2 = 2AP^2$ (Cf Ex 283)

359 Given a chord AB of a ◯, and a pt. C in it Find in the ◯ce, a pt D, such that, the st line DC shall bisect the vertical ∠ of the Δ ABD (Cal U Pap 1862, q 4)

Bisect the arc AB in E (III 30), join EC and produce EC to meet the ◯ce in D Join AD, DB, ∵ arc AE = arc EB (cons), ∴ ∠ADE = ∠BDE (III 26), ∴ D is the required pt so that, ∠ ADB is bisected by CD

360 A tangent is drawn ∥ to a chord Show that the intercepted arc is bisected at the pt of contact (Cal U Pap 1863, q 5 and 1887, q 2)

Let QR be a chord and MPN a tangent ∥ QR Let X be the centre Join QP, RP and PX cutting QR in O. Then ∠OPN is a rt ∠ (III 18), ∴ ∠POR is a rt ∠ (I 29) Hence QO = OR (III 3), and OP is common ∴ QP = RP (I 4), ∴ arc QP = arc RP (III 28)

361 To describe a ◯ that shall touch a given st line and also touch a given ◯ (Cal U Pap 1866, Cf Ex. 9, p 238 Text)

Let XY be the given st line and PMN be the given ◯ Find O the centre of the ◯ PMN (III 1) Draw OZ ⊥ XY Produce ZO to meet the ◯ce of ◯ PMN in P. Bisect PZ in S With centre S and radius SZ describe a ◯ PQZ ∵ ∠SZY is a rt ∠, ∴ the ◯ PQZ touches XY (III 16), and ◯ PQZ touches the ◯ PMN (III 11)

362. Show, by assuming the ∠ in a semi ⊙ to be a rt ∠, how III 17, may be more simply effected (Cal U Pap 1868, q 4; Vide Text p. 202)

Let X be the centre of the ◯, and Y be a pt without the ⊙ Join XY On XY describe a semi-◯ XOY cutting the ◯ce at O Join XO, OY. Then ∠XOY is a rt ∠ (III 31), ∴ touches the ◯ (III 16)

363 Two ◯s have the same centre, show that all chords of the outer ◯ which touch the inner ⊙ are equal. (Cal U Pap 1868, q 6 b).

Let XY and MN be the chords of the outer ◯ which touch the inner ◯ in the points Q and P. Let O be the common centre,

join OQ, OP Then OQ and OP are ⊥s to XY and MN respectively (III 18), but OQ=OP, being radii of the inner ◯, ∴ XY=MN (III 14)

364 If two ◯s intersect one another, their common chord when produced, bisects their common tangent (Cal U. Pap 1869, q 2.)

Let XY be the common chord, and X and Y be the points of intersection of the two ◯s, and let MN be the common tangent Produce YX to cut MN in Z, $MZ^2=XZ.ZY=ZN^2$ (III 36), ZM=ZN

365 Two st lines OA, OB, being given, intersecting in O and a pt C being given in OA, describe a ◯ touching OA in C and also touching OB (Cal U Pap 1870, q 2)

Bisect ∠AOB by OX (I 9) From C draw CX ⊥ OA (I 11), and cutting OX in X From X draw XY ⊥ OB (I. 11) In △s OCX, and OYX, ∠OYX=∠OCX, being rt ∠s, and ∠YOX = ∠COX (cons), and OX is common, XY=XC (I 26), ∴ the ◯ described with centre X and radius XC, will pass through the point Y Again OY and OC are ⊥s to YX and XC, ∴ OY and OC touch the ◯ at Y and C (III 16)

**366 If the parts of two chords at rt ∠s one another be given, explain how the* length of the radius of the ◯ may be calculated (Cal U Pap 1872, q 3)

Let MN and XY be two chords at rt ∠s to each other meeting in S Bisect MN, XY at Q and R (I 10) Find P the centre of the ◯ (III 1) Join PQ, PR, PM and PX, PQ and P are ⊥s to MN and XY respectively (III 3) Then PRSQ is a rect, ∴ PQ=RS and PR=QS Now $MS^2+SN^2=2MQ^2+2QS^2$ (II 9), and $XS^2+SY^2=2XR^2+2RS^2$ (II 9), ∴ $MS^2+SN^2+XS^2+SY^2=2MQ^2+2QS^2+2XR^2+2RS^2=2MQ^2+2PR^2+2XR^2+2PQ^2=2(MQ^2+PQ^2)+2(PR^2+XR^2)=2MP^2+2PX^2=4MP^2$ (∵ MP=PX=a radius) $=4(\text{radius})^2$, ∴ $(\text{radius})^2=\frac{1}{4}(MS^2+SN^2+XS^2+SY^2)$

367 AB is a chord of a ◯, C a point in the ◯ce of the smaller segment, find a point D in the ◯ce of the larger segment so that AB shall bisect the ∠DBC (Cal U Pap 1873, q 3)

Join AC With A as centre, and radius=AC describe a ◯ cutting the 1st ◯ in D Join AD Then D is the pt reqd. Join DB, BC, ∴ AC=AD (being radii of the constructed ◯), ∴ arc AC=arc AD, ∴ ∠DBA=∠CBA (III. 27), ∴ ∠DBC is bisected by AB

**368 If from any pt without a ⊙, st lines be drawn touching it, the ∠ contained by the tangents=2 ∠ contained by the st line joining the points of contact and the diameter through either of them* (Cal U. Pap, 1878 q 3).

Let XYZ be the ⊙, O a pt without it, OX, OY are the tangents, join XY Find P the centre of the ⊙ (III 1), join XP and YP Produce YP to meet the ⊙ce at Z Now, all the interior ∠s of the quad. XPYO=4rt ∠s (I 32, Cor), and ∠s PXO and PYO are rt ∠s (III 18), ∴ ∠s (XYP+XOY) =2rt ∠s. But ∠s (PXY+XPY+PYX)=2rt ∠s (I 32), ∴ ∠s (XPY+XOY) = ∠s (PXY + XPY + PYX), or ∠ XOY = ∠s (PXY+PYX). But ∠ PXY = ∠ PYP (I 5), since PX=PY, ∴ ∠ XOY = 2∠ PYX

**369. In two ⊙s which touch each other* externally, *two ∥ diameters are drawn Show that one extremity of each diameter and the point of contact, lie in the same st. line* (Cal U Pap, 1879. q. 1)

Let the two ⊙s touch each other at X, and let PYQ and MZN be the ∥ diameters, Y and Z being their centres Join QX, XM Then QX shall be in the same st line with XM The st line joining Y and Z shall pass through X (III 12) In ∆s YQX and ZMX, ∵ ∠ QYX= ∠ XZM (I 29), ∴ ∠YQX + ∠YXQ = ∠ ZXM + ∠ZMX (I 32), (*a*) But ∠ YQX= ∠ XQY for XY = YQ So ∠ ZXM = ∠ ZMX, hence from (*a*) above, 2 ∠ YXQ= 2 ∠ ZXM, ∴ ∠ YXQ= ∠ ZXM Add to these equals ∠ QXZ. ∴ ∠YXQ+ ∠QXZ=∠QXZ+ ∠ZXM But ∠XQY+ ∠QXZ= 2rt ∠s (I 13), ∴ ∠QXZ+ ∠ZXM = 2 rt ∠s; QX is in the same st line with XM (I 14)

370 AOC and BQD are two ∆s, having ∠ AOC= ∠ BQD, and ∠ACO=∠DBQ, show that the rectangle AO QB = the rectangle CO QD (Cal U Pap, 1880 q 2)

Produce AO, CO to Y, X respectively, making OY=BQ, and OX=QD, join XA, XY, ∵ ∆OXY=∆DQB (I 4), ∠XYO= ∠QBD, but ∠ QBD=∠ ACO, ∴ ∠ XYO=∠ACO, ∴ a ⊙ described about ∆ ACX, shall pass through Y (III. 21, *converse*), ∴ AO OY=CO OX (III 35), ∴ AO QB=CO.QD

**371 If two st lines AB, CD in a ⊙ intersect in E, the ∠s subtended by AC and BD at the center, are together=2 ∠AEC* (Cal U. Pap., 1882, Cam. U. Pap., 1843, Bom. U Pap, 1363)

Find X the centre, (III 1); join AX, BX, CX, DX and AD, ∴ ∠AXC=2 ∠ADC and ∠BXD=2 ∠BAD (III 20), ∠AXC+ ∠BXD=2 ∠ADC+2 ∠BAD But ∠ADC+∠BAD = ∠AEC (I 32), ∴ ∠AXC+∠BXD=2 ∠AEC

372 AO, BO are radii of a ◯ at rt ∠s to each other, ACD is a st line meeting OB in C, and the ◯ in D Then the rectangle AC AD is = 2 OB² (Cal. U Pap, 1882 q 5)

Produce BO to meet the ◯ in X, now XC.OB + OC² = BO² (II 5) and XC CB = AC CD (III 35), ∴ AC CD + OC² = BO² But OC² = AC² − AO² (I 47) = AC² − BO², ∴ AC CD + AC² − BO² = BO², or AC CD + AC² = 2OB² Again ∵ AD AC = AC CD + AC² (III. 3), ∴ AD AC = 2BO²

**373* Through one extremity of the common chord of two intersecting ◯s, two st lines are drawn terminated by these ◯s Prove that the st lines joining the other extremity of the common chord and the two terminal points of the two st lines on each ◯, together with the st lines joining these terminal points, form two equiangular Δs* (Cal U Pap, 1883 q 2)

Let the two ◯s cut each other in X and Y, and join XY Through Y, draw any two st lines PYQ, MYN meeting the ◯ces of each, in the pts P, M and Q, N Join XM, XP, XN and XQ Then Δ XMN shall be equiangular to Δ XPQ ∵ ∠ XMY = ∠ XPY (III 21), and ∠ XQY = ∠XNY (III 21), ∴ the rem ∠MNX = rem ∠PXQ (I 32) Hence ΔXMN is equiangular to Δ XPQ

**374 AB is a diameter of a ◯, AC a tangent at A = AB, CB is joined, cutting the ◯ in D, prove that CB is bisected in D, and AD = ½ CB* (Cal U Pap, 1885)

Since ∠ADB = a rt ∠ (III 31), ∠DAB + ∠DBA = a rt. ∠ (I 32) But ∠BAC = a rt ∠ (III 16, Cor), ∴ ∠DAB + ∠DBA = ∠ BAC Take away the com ∠ DAB, ∴ ∠ DBA = ∠ DAC But ∠ DBA = ∠ ACD, for AB = AC (hyp), ∴ ∠ ACD = ∠ DAC, hence CD = AD (I 6) Again ∠BAC = a rt ∠ (III 16, Cor), = ∠ACB + ∠ABC (I 32), and ∠DAC = ∠ACD (proved), ∴ ∠DAB = ∠ABC, ∴ AD = DB (I 6) Hence CD = AD = DB, ∴ CB is bisected at D, and AD = ½ CB

375 In III 11, show that any chord of the exterior ◯ drawn from the pt of contact, is bisected by the interior ◯, if that ◯ passes through the centre of the exterior ◯. (Cal U Pap, 1886 q. 3)

Let the two ◯s touch each other internally at the pt X, and the *inner* ◯ passes through the center Y of the *outer* ◯ From X draw any chord XPQ to the outer ◯, cutting the inner at P Then XQ shall be bisected at P Join PY, XY, ∠XPY = a rt ∠ (III 31), ∴ XQ is bisected at P (III 3)

376 Two equal ◯s intersect in A and B Let CD and EF be chords of the ◯s, each = to the chord AB, and so placed on

opposite sides of AB, that all the three chords DC, BA, FE meet in H Then AH bisects the ∠CHE (Cal U Pap, 1886 q 6)

Let X and Y be the centres of the ⊙s ABD and ABE respectively Join XY cutting AB at Z Draw XP and YQ ⊥s to CD and EF respectively Join HX, HY, XB YB, XA and YA In △s XBA, YBA, XB=YB and BA is common, XA=YA, ∴ ∠XBA=∠YBA (I 8) In △s XHZ and YHZ, ∵ XB=YB and BZ is common, and ∠XBZ=∠YBZ, XZ=ZY (I. 4), the ∠s at Z are rt ∠ In △s XZH and YZH, ∵ XZ=YZ and ZH is common and ∠XZH=∠YZH, ∴ ∠ZHX=∠ZHY (I 4) Now . $XP^2+PH^2=XH^2$ (I 47) $=XZ^2+ZH^2$ But XP=XZ (III 14), ∴ PH=ZH Hence ∠PHX=∠ZHX (I 8) or ∠PHZ=2∠ZHX Similarly ∠ZHQ=2 ∠ZHY But ∠XHZ=∠YHZ (proved), ∴ ∠PHZ=∠QHZ or AH bisects the ∠CHE

377 *AB, AC are tangents to a given ⊙, BC the chord of contact From the mid pt D of BC, st line EDF is drawn ⊥ BC cutting the ⌒ce of the given ⊙ at E and F* Prove *that E, F are the centres of two ⊙s one of which touches the three sides, and the other touches one side and two sides produced, of the △ ABC* (Cal U Pap, 1887 q 4)

(*a*) Since BC is a chord of the given ⊙, and EDF is drawn at rt ∠s to BC through its mid pt D, ∴ EDF is a diameter of the given ⊙ Join EB and BF Since EDF is a diameter, ∴ ∠EBF is an ∠ in a semi-⊙, hence ∠EBF is a rt ∠ (III 31) Now in the rt ∠d △s EBF, and EBD, ∠EBC=∠BDE=a rt ∠, ∠E is common, ∴ ∠F=∠EBD (I 32, Cor), but ∠F=∠EBA (III 32), ∴ ∠EBA=∠F=∠EBD, hence, EB bisects ∠ABC So, it can be proved that, EC bisects ∠ACB, ∴ E is the centre of the *inscribed* ⊙

(*b*) Produce AB to G Now ∠FBG=∠E (III 32), also in the rt ∠d △s EBF and DBF, ∠EBF=a rt ∠=∠BDF, and ∠F is common, ∴ ∠FBD=∠E (I. 32), hence ∠FBG=∠E =∠FBD, ∴ FB bisects the interior ∠ at B So it can be proved that FC bisects the exterior ∠ at C Hence F is centre of the ⊙ touching one side BC of the △ABC, and the other two sides AB, AC produced (*Cf Text ps 254-555*)

378 *If from a pt, two st lines be drawn to touch a ⊙, these st lines are equal (III 17, Cor p. 202, Text, or see Notes on III 17)* (Cal U. Pap, 1888 q. 3).

379 *The three ⊥s dropped from the vertices on the opposite sides of any △, meet at a point (See Text p 106, Ex. 6, and p. 224 Ex 19)* (Cal. U. Pap, 1889 q 5 and 1895 q. 9)

380 If two opposite sides of a quadrilateral inscribed in a ◯ are equal, prove that the other two sides are ||| (Cal U Pap, 1890 q 5)

Let ABCD be a quadrilateral inscribed in a ◯, and it is given that AB=CD, then shall AD be ||| to BC Join AC Since chord AB=chord CD ∴ arc AB=arc CD (III 28), hence ∠ACB=∠CAD (III 27), and these are *alternate* ∠s, AD is ||| to BC

381 P is a pt in arc APB, of a ◯, the tangent at P meets the chords AB produced in R, and meets AQ (the ⊥ to AB) in Q If QR is bisected in P, prove ∠ABP=2∠BAP (Cal U Pap., 1890 q 6)

Join AP, PB, ∵ ΔQAR is a rt ∠d Δ, and P is the mid pt of its hypotenuse, PA=PR (*Ex* 107), ∠*PAR*=∠PRA (I 5), but ∠BPR=∠*PAR* (III 32), ∠PRA=∠BPR, for each=∠PAR or ∠*PAB*, ∠PRA+∠BPR=2 ∠PAB, also ∠*ABP* ∠PRA+∠BPR (I 32, Case II), ∠ABP=2∠PAB

382 Draw a common tangent to two ◯s, *and shew that in general,* four *common tangents may be drawn to* two *given* ◯*s* (Text ps 218-219, Ex 17) (Cal U Pap, 1891 q 3)

383 Describe a ◯ to touch a given ◯ and also to touch a given st line at a given pt (Text Ex 39, p 221 (Cal U Pap, 1892 q 4)

384 Of all Δs of given base *and* area, *the isosceles is that which has the* least *perimeter* (Cal U Pap, 1892 q 5)

Let MN be the given base, and A the *vertex* of all Δs having the base MN, and *area given* Now if the *base* and *area* of *one* of these Δs ABC be given, the vertex A must be on a st line XY ||| to the given base MN Since MN the base is *given,* the perimeter of the Δ is *least* when (AM+AN) is *least* Now from (*Note to Ex 3, p 243 Text*), we know that (AM+AN) is *least,* when AM, and AN are *equally inclined* to XY *i e* when ∠XAM=∠YAN, but since XY is ||| to MN and AM, AN meets them, ∠*XAM*=∠AMN, and ∠*YAN*=∠ANM, but ∠XAM=∠YAN, ∠AMN=∠ANM, AM=AN, Δ AMN is isosceles, which has the *least* perimeter

385 Two ◯s touch each other externally *at A, and a st. line touches them in B, C Prove that ∠BAC=a rt. ∠* (Cal U Pap, 1893 q 3)

Draw AO another *common tangent* to the ◯s (Ex 382) ∴ OB=OA (III 17, Cor), ∴ ∠OBA=∠OAB, and ∵ OC

=OA, (III 17, Cor), ∴ ∠OCA=∠OAC, ∴ ∠B+∠C=∠BAC *i e* (one ∠ of a △= sum of the other two ∠s), ∠BAC=a rt ∠, (I 32)

386 Given the base and vertical ∠ of a △, find the locus *of the centre of the inscribed ○* (This is the same as Ex 36, p 228, Text) (Cal U Pap, 1893 q 4).

387 Describe a ○, passing through two given pts, and touching a given st line (Text p 235, Ex 21), (Cal U Pap, 1894 q 9)

** 388 Being given two intersecting st lines and a point O, you are required to draw through O, a st line meeting the given st lines in P and Q, so that the rect OP OQ may be* given *i e =a given rectangle.* (Cal U. Pap, 1895 q 10)

Let AB and AC be the given intersecting st lines meeting in A *Two cases may arise* The pt O may be (within) the ∠BAC, or (outside) it

(*a*) *Case I First*, take O within the ∠ BAC.

Join AO, and produce AO to E, so that rect AO OE=the *given rectangle* On OE, describe a segment of a ○ containing an ∠ = ∠ BAO (III 33); let this segment cut AC in Q Join QO, and produce it to meet AB in P, then *PQ is the st. line reqd.* For ∠ *PAE* or ∠ BAO=∠ *PQE* (const), ∴ points P, A, Q, E are *concyclic* (III 21, *Con*) Hence rect *OP OQ*=rect AO OE (III 35)=the given rectangle

(*b*) *Case II*, when O is without the ∠BAC

Join AO, and on AO, take E so that AO OE=the *given rectangle*, and on OE, describe a segment of a ○ containing an ∠ = ∠ OAB (III 33), let this segment of the ○ cut AC in Q Join QO meeting AB in P Then OPQ is the reqd st line For ∠*EAP* or ∠ OAB= ∠ EQP (const), ∴ points E A, Q, P are *concyclic* (III 21, *Con*), hence rect OP OQ=rect OA OE (III 36) =the given rectangle

389 Two ○s intersect in the pts P and Q Show that the st line joining the centres of the two ○s cuts PQ at rt ∠s (Bom U Pap, 1859) *(See Ex 25, or Ex 306)*

390 (Old No. 378) Upon a given base, describe an isosceles △, whose vertical ∠ shall be=½ of the vertical ∠ of a given isosceles △, upon the same base (Bom U Pap, 1863)

Let XYZ be an isosceles △ standing on the base YZ, it is reqd to draw another isosceles △ on the same base YZ, having its vertical ∠ =½ of the vertical ∠ YXZ From the centre X at

the distance XY or XZ, describe a ○ YZP which will pass through Y, Z. From X draw XO ⊥ to YZ (I. 12). Produce OX to meet the ○ce in P, join PY and PZ. Then PYZ is the required Δ. Now Δ XYO = Δ XZO (I. 26), and YO = ZO (III. 3). Again ∵ YO = ZO and OP is common and ∠ YOP = ∠ ZOP (rt. ∠s), ∴ YP = ZP, ∴ Δ PYZ is isosceles. Again the vertical ∠ YPZ = ½ ∠ YXZ (III. 20). ∴ Δ YPZ is described on the base YZ, having the ∠ YPY = ½ ∠ YXZ.

391. (*Old No. 379*). Draw two concentric ○s such that those chords of the outer ○ which touch the inner, may be = to its diameter. (*Bom. U. Pap.*, *1865*.)

Take a st. line XY, at Y make an ∠ XYZ = ½ a rt. ∠ (I. 23). From the centre Y at the distance YX, describe a ○ XPQ cutting YZ in P, through X draw a tangent RXZ (III. 17) meeting YP prod. in Z, from the centre Y with the radius YZ, describe a ○ RZS, then RXZ shall be = the diameter of the inner ○ XPQ. ∵ ∠ YXZ = a rt. ∠ (III. 18), and ∠ XYZ = ½ a rt. ∠ (Cons.), ∴ ∠ XZY = ½ a rt. ∠ (I. 32), ∴ XZ = XY (I. 6). But XZ = ½ RZ, for YX bisects RZ (III. 3) and XY = ½ of the diameter of the inner ○, ∴ the chord RZ = the diameter of the inner ○.

392. (*Old No. 380*). In the diameter of a ○ produced, determine a pt. so that the tangent drawn from it to the ○ce, shall be of given length. (*Bom. U. Pap.*, *1873*.)

Let XYZ be a ○, whose centre is O, and the diameter XY is produced to P. From the centre O, draw OQ at rt. ∠s to XP, meeting the ○ce at Q. Join OP cutting the ○ce at Z, join OZ, and at Z draw ZR at rt. ∠s to OZ (I. 11), intersecting YP at R. RZ is evidently a tangent to the ○ (III. 16). Then R will be the req. pt. For in the Δ OQP, since ∠ QOP is a rt. ∠, ∴ ∠ OQP + ∠ OPQ = a rt. ∠ (I. 32). Also ∠ OZR = a rt. ∠, ∴ ∠ OZQ + ∠ RZP = a rt. ∠, ∴ ∠ OZQ + ∠ RZP = ∠ OQP or ∠ OQZ + ∠ OPQ. But OZ = OQ, ∴ ∠ OQZ = ∠ OZQ, ∴ ∠ RZP = ∠ OPZ ∴ RP = RZ (I. 6), ∴ R is the pt. reqd. from which tangent RZ drawn to the ○ = a length RP.

393. (Old No. *381*). AB is a diameter of a ○, C any point in its ○ce, AC, BC produced meet the tangents at B and A,—in D and E, and the tangent at C meets the same tangents in F, G, show that FG = ½ (BD + AE). (*Bom. U. Pap.*, *1878*.)

Since FC = FB (III. 17, Cor.), ∠ FCB = ∠ FBC (I. 5), but ∠ FCB = ∠ *FCG* (I. 15), and ∠ FBC = ∠ *CEG* (I. 29), ∴ ∠ ECG = ∠ CEG, ∴ GE = GC (I. 6), also GC = GA (III. 17, Cor.), ∴ GC = ½AE, so CF = ½ BD, ∴ FG = ½ (BD + AE).

394. (Old No 382) Two equal ⊙s touch each other externally ; *and through the pt of contact, chords are drawn one to each ⊙ at rt. ∠s to each other ; prove that the st line joining the other extremities of these chords, is = and ∥ to the st. line joining the centres of the ⊙s* (Bom U Pap., 1880, and Cam U Pap. 1859)

Let the two equal ⊙s XYZ, PXQ having their centres O and S respectively, touch each other *externally* in X, and let the two chords XP, XY be at rt ∠s to each other, then YP shall be = and ∥ to OS Join OS, YO, PS, then ∵ ∠YXP = a rt ∠ (hyp), ∴ ∠XYP + ∠XPY = a rt ∠ (I 32) So ∠YXO + ∠PXS = a rt ∠ (I. 32) But ∠OXY = ∠OYX and ∠SXP = ∠SPX (I 5), (∠OYX + ∠XPS) + (∠XYP + ∠XPY) = 2 rt ∠s, hence OY, SP are ∥ to one another (I 28), also OY = SP, for they are radii of equal ⊙s ; OY, SP are both = and ∥ st lines, and they are joined by YP and OS, ∴ YP, OS are both = and ∥ (I 33)

395 (Old No 383) If AD, CE be drawn ⊥ to the sides BC, AB of the △ ABC, and DE be joined, prove that ∠ADE = ∠ACE (Bom U Pap, 1881)

Bisect AC at X (I 10), and join EX, DX, since EX, DX are drawn from the rt ∠s at E and D to the middle pt. of the hypotenuse AC of the △s AEC ACD, ∴ each of them, (*viz* CX, DX) = $\frac{1}{2}$ AC, ∴ AX = CX = EX = DX From the centre X with any one of the four st lines as radius, describe a ⊙ ACDE, then ∠ADE = ∠ACE, for they stand upon the same arc AE (III 21)

396. (Old No 384) In a ⊙, the extremities of two radii at rt ∠s to each other, are joined. Prove that the ∠ in the segment so formed, is = $\frac{3}{2}$ rt ∠ (Bom U. Pap, 1882).

In the ⊙ XYZQ, the centre of which is P, if XP, PZ be two radii at rt ∠s to each other, the ∠ZQX in the segment XQZ formed by joining ZX, shall be = a rt ∠ + $\frac{1}{2}$ a rt ∠ Produce XP to meet the ⊙ce at Y, ∴ XY is a diameter, join QY, then ∵ ∠ at P is a rt. ∠, ∴ ∠PXZ + ∠PZX = a rt ∠ (I 32), and these ∠s are equal (I 5) ; ∴ ∠ZXP = $\frac{1}{2}$ a rt ∠ But ∠ZXY or ∠ZXP = ∠ZQY (III 21), ∴ ∠ZQY = $\frac{1}{2}$ a rt. ∠ And since ∠XQY = a rt ∠ (III 31), ∴ ∠ZQX = a rt ∠ + $\frac{1}{2}$ a rt. ∠

397 (Old No 385) Two tangents are drawn to a ⊙ at the opposite extremities of the diameter, and cut off from the third tangent a portion AB, if C be the centre of the ⊙—show that ∠ACB is a rt ∠ (Bom. U Pap, 1883)

Let YCZ be the diameter of the ⊙, and let the two tangents YA, ZB cut off from a third tangent a portion AB, then ∠ACB shall be a rt ∠. Join the pt. of contact O of the 3rd tangent with

the centre C Since $AO = XY$ (III 17, Cor), and AC is com and $CO = CY$, ∴ $\angle YAC = \angle OAC$ (I 8), and $\angle YCA = \angle OCA$. So $\angle OCB = \angle ZCB$ (I 8) But $\angle OCY + \angle OCZ = (\angle YCA + \angle ACO) + (\angle OCB + \angle BCZ) = 2 \angle ACO + 2 \angle OCB = 2 (\angle ACO + \angle OCB) = 2 \angle ACB = 2$ rt $\angle$s, ∴ $\angle ACB =$ one rt $\angle$

398 *(Old No 386) Show that all equal st lines in a ◯, will be touched by another ◯* (Cam U Pap, 1848)

Let XY, PQ be any two equal chords in a ◯ Then they are equidistant from S the centre (III 15) Draw ⊥s SN, SM to XY, PQ Then a ◯ described with the centre S and distance SM will touch each of the equal chords

399 *(Old No 387) A given st line is drawn at rt $\angle$s to the st line joining the centres of two given ◯s, prove that the difference between the squares on two tangents drawn, one to each ◯ from any point on the given st line, is constant* (Cam U Pap, 1865)

Let XY be ⊥ to PQ the st line joining the centres P and Q of two given ◯s Draw XO, XZ tangents to the ◯s (III 17), join PO, XP, XQ, QZ Now $XO^2 = PX^2 - PO^2$ (I 47), $XZ^2 = XQ^2 - QZ^2$ (I 47), ∴ $OX^2 - ZX^2 = (PX^2 - PO^2) - (XQ^2 - QZ^2) = (PX^2 - QX^2) + (QZ^2 - PO^2)$ Now $PX^2 - QX^2 = PY^2 - QY^2$ (I 47) = a constant And $QZ^2 - PO^2$ = a constant, ∴ $OX^2 - ZX^2$ is a constant

400 *(Old No 388) Describe a ◯, which shall touch a given st line at a given pt, and bisect the ◯ce of a given ◯* (Cam U Pap, 1858)

Let O be the centre of the given ◯, X the given pt in the st line YZ Draw XP ⊥ to YZ, join XO and produce it to Q, so that rect XO OQ = *sq on radius of given ◯* Then Q is known, and it is a pt in the ◯ce of the ◯, which has to be described Bisect XQ at S and draw SR ⊥ to XQ, meeting XP in R Then R will be the centre of the ◯ passing through Q and touching YZ in X Since the rect XO OQ = *sq on radius of original ◯*, the common chord of the two ◯s will evidently pass through O, and the new ◯ will therefore bisect the original ◯

401 *(Old No 389) If the opposite sides XY, ZS, and XS, YZ of a quadrilateral XYZS inscribed in a ◯, be produced to meet in P, Q, respectively, and about the △s XSP and SZQ so formed,* **without** *the quadrilateral,—◯s be described meeting again in R, shew that P, R, Q will be in one st line* (Cam U Pap, 1854).

Let the ⊙s described about XSP, SZQ meet in R Then $\angle PRS$ = supplement of $\angle PXS$ (III 22) = $\angle YXS$ (I 13) And $\angle QRS$ = supplement of $\angle QZS$ (III 22) = $\angle YZS$ (I 13.) Now

∠YXS = ∠YZS = 2 rt ∠s (III 22); ∴ ∠PRS + ∠QRS = 2 rt ∠s ∴ PRQ is a st line (I 14)

*402 *(Old No 390) Prove that the sum of the ∠s in the four segments of the ◯, exterior to the quadrilateral inscribed in a ◯ = to six rt ∠s* (Cam U. Pap, 1857)

Let XYRP be a quadrilateral, and take O, Z, Q, S pts in the *four segments exterior* to the quadril Join OX, OY, SX, SP; QP, QR ZY ZR Then ∠XOY + ∠YZX = 2 rt ∠s (III 22), and ∠XZP + ∠XSP = 2 rt ∠s (III 22), also ∠PZR + ∠PQR = 2rt ∠s (III 22), ∴ (∠XOY + ∠*YZX* + ∠*XZP* + ∠XSP + ∠*PZR* + ∠PQR) = ∠XOY + (∠YZX + ∠XZP + ∠PZR) + ∠PQR + ∠PSY, ∴ ∠XOY + YZR + ∠PQR + ∠PSX = six rt ∠s

403 *(Old No 391) If the chords, which bisect two ∠s of a △ inscribed in a ◯, be equal, prove that either one pair of its ∠s are equal, or its third ∠, is = to the ∠ of an equilateral △* (Cam U Pap, 1867)

Case I If the bisectors pass through the centre

Let PQ, XY be *equal chords passing through the centre of the* ◯, bisect two ∠s of the △ PXZ inscribed in it Now ∠QXP = ∠YQX subtended by equal chords (III 27-28), ∴ ∠YQX + ∠QXP = 2 rt ∠s (III 31), ∴ YQ is ∥ PX (I 28) ∴ ∠*QPX* = ∠YQP (I 29) = ∠*YXP* (III 21), and ∴ ∠ZPX = 2∠QPX = 2∠YXP = ∠ZXP

Case II If the bisectors be on opposite sides of the centre

∠YPX = ∠PZQ (*a*) (subtended by equal chords) But ∠YPX = ∠ZPX + ∠*YPZ* = + ∠ZPX + ∠*YXZ* (III 21) = ∠ZPX + ½ ∠*ZXP* (hyp), ∴ ∠YPX = ∠ZPX + ½∠ZXP (1). And ∠PZQ = ∠PZX + ∠*XZQ* = ∠PZX + ∠*XPQ* (III 21) = ∠PZX + ½ ∠*ZPX*, (hyp) ∴ ∠PZQ = ∠PZX + ½∠ZPX (2) From (*a*), (1) and (2), we have ∠PZX + ½ ∠ZPX = ∠ZPX + ½ ∠ZXP (*Principal Equation*); ∴ taking ½ ∠ZPX from both sides, we have ∠PZX = ½ ZPX + ½∠ZXP = ½ (∠ZPX + ∠ZXP) or 2∠PZX = ∠ZPX + ∠ZXP (3), adding ∠PZX to both we have, 3 ∠PZX = ∠YPX + ∠ZXP + ∠PZX = 3 ∠s of the △ZPX = 2 rt ∠s (I 32), hence ∠PZX = ⅓ of 2 rt ∠s = ⅔ rt ∠ = ∠ of an equilateral △.

404. *(Old No 392)* If any number of △s, upon the same base BC, and on the same side of it, have their vertical ∠s equal, and ⊥s meeting in D be drawn from B, C upon the opposite sides—find the locus of D, and shew that all the st lines which bisect the ∠BDC, pass through the same point *(Cam U Pap, 1855)*

Let BOC be any one of the △s on the base BC, draw BDX

Construct an equilateral $\triangle XYZ$, and draw $XP \perp$ to YZ. (I 12) Then $\angle YXP = \frac{1}{6}$ of 2 rt $\angle$s Draw MON a tangent to the ⊙ QOS (III 17) Make $\angle SON = \angle YXP$ (I 23) $= \frac{1}{6}$ of 2 rt $\angle$s, $\therefore \angle SOM$ (supp of $\angle SON$) $= \frac{5}{6}$ of 2 rt $\angle$s, $\angle$ in segment SRO $=$ *five times* $\angle$ in segment SQO (III 32)

412. (Old No 400) Through a pt within a ⊙, draw a chord, such that the rect contained by the whole chord and one part may be = a given square Determine the necessary limits to the magnitudes of this square (Cam U Pap, 1868)

Let O be the point within the given ⊙, whose centre *the same*
YXY′ its diameter through O, and let AB be the side *YZ are in*
square By (Ex 164), we can make a square *vertex with the*
Let $MN^2 = XY^2 - XO^2$ But $XY^2 - XO^2 =$ *the base, produced if*
(II 5, Cor) $= (XY' + XO)\,YO = OY'$ *two positions* (Cam U
$AQ = MN$, on AB as diame
centre and AQ as radius, *position of the $\triangle XYZ$ when it has been*
join AD, BD Now th *and Y′Z′)* Let the two positions of the
$\therefore AB^2 = AD^2 +$ ect in P Join XP Then $\angle XZ'P =$ supplement
OY′ YO (2) Wit = supplement of $\angle XZP$ (hyp), $\therefore$ a ⊙ can be
the given origin *XZPZ′* (III 22 *Con*), and since chord XZ = chord
of the o $\angle XPZ' = \angle XPZ$ (III 28)
OS DB² (3)
+ OY′ (*Old No 394) Two ⊙s intersect in A, B, PAP′,*
drawn equally inclined to AB to meet the ⊙s in P P′,
Prove that PP′ is = QQ′ (Cam U Pap, 1872)
be > th

ugh B draw XBY meeting the ⊙s in X, Y Join P′Y P′B, QB, PB Then $\angle P'YB =$ supplement of $\angle BAP'$ (III 22) $\angle BAP$ (I 13) $= \angle Q'AB$, (PAP′ and QAQ′ are equally inclined to AB), $\therefore P'B = Q'B$ (III 28, 29) So $PB = QB$, also Q $= \angle PAQ$ (III 21) $= \angle P'AQ$ (I 15) $= \angle P'BQ'$ (III 21), ng $\angle QBP'$ to each, we have $\angle PBP' = \angle QBQ'$ Hence in $\triangle$s PBP′ QBQ′, PB = QB and P′B = Q′B and $\angle PBP' = \angle QBQ'$, $\therefore PP' = QQ'$ (I 4)

407 (Old No 395) Through a given pt without a ⊙, draw a chord such that the difference of the ∠s in the two segments, into which it divides the ⊙, may be = to a given ∠ (Cam U Pap, 1868)

Let P be the given pt, XYZ be the ⊙, $2x$ the given $\angle$ Draw PXN a tangent to the ⊙ at X (III 17), and make $\angle PXY =$ a rt $\angle - x$ Then $\angle$ in the segment XZY = a rt $\angle - x$ (III 32), difference of the $\angle$s in the segments XMY, XZY $= 2x$ Then if in ⊙XYZ a *concentric* ⊙ be described touching XY, and a tangent to the inner ⊙ be drawn from P, the part of the tangent intercepted by ⊙XYZ will cut off the reqd segments

∵ ∠ABC = ∠AXC (III 21), ∴ ∠AEP = ∠AXC, but ∠AXC + ∠CXP = ∠AEP or ∠ CEP + ∠ CXP, . a ◯ may be described about C,E,P,Z (*Con* of III 22), ∠YCE + ∠XPE = 2 rt ∠s (III 22), and ∠XCE is a rt ∠, since AX is a diameter (III 31), ∠XPE is a rt ∠, ∴ AO or AP is ⊥ to DE

415 (Old No 403) If from a given pt A, without a given ◯, any two st lines APQ, ARS be drawn, making equal ∠s with the diameter, which passes through A, and cutting the ◯ in P, Q and R, S respectively, then PS QR shall cut one another in a of ∠CAZ (Cam U Pap, 1864)

+ ∠XED + the centre of the ◯, XO its diameter It is evident + ∠ DEA) (I intersect on some pt Z in AY Join OR, OP and ∠DEA is common, . = ∠PQR (III 20), ∴ P, Z, O, Q can be + ∠DAB) = complement of (on), rect AZ AO = rect AP AQ ∠DAF = complement of ∠ CAF (Cor), ∴ Z is a fixed pt (a) and (b), XE = XF, must of the s e, on AB produced

Case II When AEF bisects the exterior ∠m the vertex X *DA*, the proof is similar se, or the base

409 (Old No 397) If four s be drawn through three out of four given points, the ∠ bet gents at the intersection of two of the s, is = to the tangents, at the intersection of the other two ◯s (Apply III 3) all st lines ated by the ne join- Pap 1866)

Let XQ, XQ be tangents to the ◯ passing through XYP Then ∠ QXP = ∠ XYP (III 32), and Q′XZ = ∠ (III 32), ∠ QXP + ∠Q′XZ = ∠XYP + ∠XYZ, to each, ∠ZXP, ∠ Q′XQ = ∠ZXP + (∠XYP + ∠XYZ) = ∠ZXP ∠ZYP, So the ∠ between the tangents to the ◯s passing throu X,Z, P and Z, Y, P = ∠XZY + ∠XPY = 4 rt ∠s *diminished* by su of ∠s ZXP, XYP, hence the acute ∠s between them, are the same in both cases Let and AB to Q n

410 (Old No 398) AB,CD are ||| diameters *of two* ◯s, *and AC cuts the* ◯*s in P, Q, prove that the tangents to the* ◯*s at P,Q—are* ||| (Cam U Pap, 1870)

Let YP and XQ be the tangents of two ◯s at P and Q, join PB, QD Then ∠YPA = ∠ABP (III 32) = complement of ∠PAB (III 31) = complement of ∠QCD, (since AB, CD are ||| and AC meet them) = ∠ QDC (III 31) = ∠*XQP* (III 32), YP is ||| XQ (I 28)

411 (Old No 399) Divide a ◯, *into two segments, such that the* ∠ *in one of them, shall be* five *times the* ∠ *in the other.* (Cam U Pap, 1850)

Construct an equilateral ΔXYZ, and draw XP $\perp$ to YZ (I 12) Then $\angle$ YXP = 1/6 of 2 rt $\angle$s Draw MON a tangent to the ◯ QOS (III 17) Make $\angle$ SON = $\angle$ YXP (I 23) = 1/6 of 2 rt $\angle$s, ∴ $\angle$ SOM (supp of $\angle$ SON) = 5/6 of 2 rt $\angle$s, ∴ $\angle$ in segment SRO = *five times* $\angle$ in segment SQO (III 32)

412. (Old No 400) Through a pt within a ◯, draw a chord, such that the rect contained by the whole chord and one part may be = a given square Determine the necessary limits to the magnitudes of this square (Cam U Pap, 1868)

Let O be the point within the given ◯, whose centre is X, and YXY′ its diameter through O, and let AB be the side of the given square By (Ex 164), we can make a square = diff of two sq Let $MN^2 = XY^2 - XO^2$ But $XY^2 - XO^2 = (XY + XO)(XY - XO)$ (II 5, Cor) = (XY′ + XO) YO = OY′ OY (1) From AB cut off AQ = MN, on AB as diameter, describe a semi-◯, with A as centre and AQ as radius, describe a ◯ cutting the semi-◯ at D, join AD, BD Now the $\angle$ ADB in the semi-◯ = a rt $\angle$ (III 31), ∴ $AB^2 = AD^2 + BD^2$ (I 47) Also $AD^2 = MN^2 = XY^2 - XO^2 = OY' YO$ (2) With centre O and radius BD describe a ◯ cutting the given original ◯ in S, join OS, produce SO to meet the ◯ of the original ◯ in P Now OS = DB (being radii), ∴ $OS^2 = DB^2$ (3) Then rect PS SO = OS^2 + PO OS (II 3) = OS^2 + YO OY′ (III 35) = $DB^2 + AD^2 = AB^2$ = the given square

N B The *limits* are deducible from the fact, that DB must be > than OY and < than OY′

413 (Old No 401) From a given pt O as centre, describe a ◯ cutting a given st line XYZ in two pts Z, Y, so that the rect contained by their distances from a fixed point X in the given st line XYZ, may be = a given square (Cam U Pap 1853)

Describe a semi-◯ on XO as diameter, and with X as centre, and radius = *a side of the given square*, describe a ◯, cutting the semi-◯ in P Join XP, which is = side of given square With centre O and radius OP, describe another ◯ cutting the *given st line* in Y, Z This ◯ touches XP (III 16), ∵ $\angle$ XPO = a rt $\angle$ (III 31), ∴ rect XY XZ = XP^2 (III 36) = a given square

414 (Old No 402) If two chords AB, AC, be drawn from any pt A of a ◯, and be produced to D and E, so that the rect AC AE = rect AB AD, then, if O be the centre of the ◯,—AO is ⊥ to DE. (Cam U Pap, 1860)

Join AO and produce it to meet DE in P, cutting the ◯ in X Since AC AE = AB AD, a ◯ may be described about C, B, D, E (III 36, Cor), ∴ $\angle$ *ABC* + $\angle$ CBD = 2 rt $\angle$s (I 13) = $\angle$ CBD + $\angle$ CEP or $\angle$ *AEP* (III 22), ∴ $\angle$ ABC = $\angle$ AEP Join CX Then

∵ ∠ABC = ∠AXC (III 21) ∴ ∠AEP = ∠AXC, but ∠AXC + ∠CXP = ∠AEP or ∠CEP + ∠CXP ∴ a ⊙ may be described about C, E, P, Z (*Con* of III 22), ∠XCE + ∠XPE = 2 rt ∠s (III 22), and ∠XCE is a rt ∠, since AX is a diameter (III 31) ∠XPE is a rt ∠, ∴ AO or AP is ⊥ to DE

415 (*Old Nb* 403) *If from a given pt A, without a given* ⊙, *any two st lines APQ, ARS be drawn, making equal ∠s with the diameter, which passes through A and cutting the* ⊙ *in P, Q and R, S respectively, then PS, QR shall cut one another in a given point.* (Cam. U Pap, 1864)

Let O be the centre of the ⊙, XOY its diameter It is evident that PS, QR, intersect on some pt Z in XY Join OR, OP and $\angle POA = \frac{1}{2}$ ∠POR (I 8) = ∠PQR (III 20), ∴ P, Z, O, Q can be circumscribed by a ⊙ (III 21, *Con*) rect AZ AO = rect AP AQ (III 36, Cor) = rect. AX AY, (III 36, Cor), ∴ Z is a fixed pt

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416 XYZ is an isosceles △, and from the vertex X as centre, a ⊙ is described cutting the base or the base produced, at P and Q, shew that YP = ZQ. (*Apply III 3*)

417 If two ⊙s cut one another, of all st lines drawn through a pt of section and terminated by the ⊙ces, the greatest is that which is ∥ to the st line joining the centres

Let the two ⊙s whose centres are P and Q cut at O, X Let AOB be the st line through O ∥ to PQ terminated by the ⊙ces, and let COD be any other st line terminated by the ⊙ces *Then AB shall be > CD* By drawing ⊥s PM and QN from P, and Q to AB, it is seen that AB = 2MN (III 3) = 2 PQ (I 34) From P, Q draw PZ, QY ⊥s to CD, and from Q draw QE ⊥ to PZ Then CD = 2ZY (III 3) = 2EQ (I 34) But in the rt ∠d △ EPQ, PQ > EQ, (since hypotenuse to the greatest side), ∴ 2PQ > 2EQ or 2 ZY, AB > CD

418 Through a pt of section of two ⊙s which cut one another, draw a st line terminated by the ⊙ces, and bisected at the pt of section

Let O, M be the pts of section of the ⊙s whose centres are X and Y; join XY and bisect XY at D Join OD and draw ZOA ⊥ to OD and terminated by the ⊙ces *Then shall ZO be = AO*, draw XB and YC ⊥s to ZA Since XB, DO, YC are all ∥, and XD = YD (cons), ∴ BO = CO, hence ZO = AO (III. 3).

419 Shew that the st lines drawn at rt. ∠s to the

sides of a quadrilateral inscribed in a ○, from their middle points—intersect at a fixed point

Let MN, SN, RN, QN bisect ⊥ly the sides of the quadrilateral XYZP inscribed in the ○ XYZ Then N the intersection of those lines, *shall be a fixed pt* Since QN bisects XP ⊥ly, the centre of the ○, lies in QN, so the centre of the ○ lies in each of the st lines RN, SN, MN (III 1, Cor) Hence the pt N, where they intersect, is the centre of the ○, and ∴ a fixed pt

420 If two ○s cut one another, any two ||l st lines drawn through the pts of intersection and terminated by the ○ces,—are equal

Let P, Q be the pts of intersection, O, S the centres of two ○s, draw ||l st lines APB, CQD cutting the ○ce of the ○ with centre O, at A, C, and the ○ce of the ○ with centre S, at B, D Draw st lines XOY and MSN, through O and S (the centres), ⊥s to AB and CD, meeting AB at X, and M, and CD at Y and N Now, it is obvious that, XYNM is a *rectangle*, XM = YN (I 34) (*a*) We know that PX = XA, and PM = MB (III 3), ∴ AB = 2XM (1) So, it can be proved that CF = 2 YN (2), ∴ from (1), (2) and (*a*), AB = CD

421 If two ○s cut one another, any two st lines drawn through a pt of section, making equal ∠s with the common chord, and terminated by ○ces, are equal

Let the two ○s intersect at P, Q, and let APB and CPD be two st lines equally inclined to PQ and terminated by the ○ces Through Q draw EQF ||l AB (I 31), since AB is ||l EF, and PQ meets them, ∠*PQE* = ∠QPB (I 29), but ∠QPB = ∠QPC (hyp), ∴ ∠PQE = ∠QPC (Ax. 1), ∴ arc PACE = arc CEQ (III 26), take away the common arc CE from both, ∴ rem arc PAC = rem arc QE, ∴ chord PC = chord QE (III 29) (1) So, can be proved that chord PD = chord QF (2) From (1) and (2), *EF* = CD, but *EF* = AB (*Ex* 420), ∴ CD = AB

422 Two equal ○s touch one another externally; and through the pt of contact, two chords are drawn, one in each ○, at rt ∠s to each other, shew that the st line joining their other extremities, is = to the diameter of either ○

Let P, Q be the centres of two equal ○s, which touch one another *externally* at O, and from O, let chords OX, OY be drawn at rt ∠s to each other Join PX, QY. Then PQ passes through O (III 12), and ∠ POX + ∠ QOY = one rt ∠ (I 13), ∴ (∠POX + ∠PXO) + (∠QOY + ∠QYO) = 2 rt ∠s, ∴ ∠XPO + ∠ YQO = 2 rt ∠s (I 32), ∴ XP is ||l YQ and XP = YQ, (being radii of equal ○s), ∴ XY ||l and = to PQ

423 All equal chords placed in a given circle, touch a fixed concentric ○

Let XY be a chord of *fixed length* M its mid pt, and Z the centre of the ○ Join ZM Then ZM is ⊥ to XY (III 3), and is of *fixed length*, for all positions of XY (III 14), since XY is ⊥ to the radius ZM (III 16), ∴ the *locus* of M is a concentric ○, which is touched by XY at M

424 If a quadrilateral XYZP be circumscribed about a ○, then prove that XY + PZ = XP + YZ

Let the pts of contact of the sides XY, YZ, ZP, PX being E, F, G, H, respectively Now *XE* = XH, *YE* = YF, *ZG* = ZF, *PG* = PH (III 17, Cor) Adding, {(XE + YE) + (ZG + PG)} = {(XH + PH) (YF + ZF)} or XY + PZ = XP + YZ

425 If the sum of one pair of opposite sides of a quadrilateral XYZP is = to the sum of the other pair, i e XY + PZ = PY + ZY, shew that a ○ may be inscribed in the figure

Bisect ∠s XYZ and YZP by YA and ZA meeting at A From A draw AE, AF, AG ⊥s to XY, YZ and ZP It is obvious that AF = AE = AG Now, if with A as centre and radius = AE, or AF, or AG a ○ be described, then that ○ will touch XY, YZ and ZP in E, F, G Then shall XP touch also this ○ For if not, from X draw XO touching the ○ (III 17), and meeting PZ at O Now XY + PZ = XP + YZ (Hyp) (1) Also XY + ZO = YZ + XO (Ex 424), (2) ∴ taking (2) from (1) PZ − ZO or *PO* = XP − XO, hence either XO = PO + XP, or XP = XO + PO, which is impossible (I 20)

426 St lines are drawn from a given external pt O, to the ○ce of a ○, find the locus of their mid pts

Let X be the centre of the ○, B be any pt on the ○ce Join OB, and let A be the mid pt of OB The *locus* of A is reqd Join XO Bisect XO at P, and join PA Then ∵ P, A are the middle pts of OX, OB, ∴ AP = ½ XB (the radius of the ○), (Text Ex 2, p 96), *i e*, PA is of *constant length*, and P is the fixed pt Hence the *locus* of A, is a ○, whose radius = ½ the radius of the given ○

427 Find the locus of pts, such that the pairs of tangents drawn from them, to a given ○ contain a constant ∠

Let XY, XZ be any *pair of tangents* containing the given ∠ Let O the centre of the given ○, join OZ, OX Now XO bisects the ∠ YXZ (III 17, Cor) Hence in the Δ OXZ the ∠s OXZ,

OZX, and the side OZ are *constant*, OX is constant, ∴ the locus X is a concentric ◯ with OX as radius

428 Two ◯s whose centres are X and Y, have external contact at P, and a direct common tangent RS is drawn to touch them at R and S. Shew that the bisectors of the ∠s RXP, SYP meet at rt ∠s in RS; and if O be the point of intersection of the bisectors, shew that OP is also a common tangent to the ◯s

Let the bisector of the ∠ RXP meet RS at O Join OP Then △ XRO = △ XPO (I 4), ∴ ∠OPX is a rt ∠ Hence OP is the tangent to both ◯s at P (III 16) Thus the bisector of the ∠RXP, meets RS at the pt at which it is cut by the tangent at P So the bisector of the ∠ SYP meets RS at the same pt *i e* the bisectors intersect on RS and at rt ∠s, for they are also the bisectors of the ∠s ROP, SOP

429 Through a given pt within a ◯, draw the *shortest* chord (*See General notes on III 15*)

430 Through a given pt, draw a st line to cut a ◯, so that the part intercepted by the ◯ce, may be = to a given st line

Between what *limits* must the given st line lie, when the given point is (*a*) *without* the ◯, and (*b*) *within* it?

Let X be the given pt, P the centre of the given ◯ and Y be the given st line In the given ◯, take any diameter SPMN, from it cut off SM = Y (I 3) With S as centre and SM as radius, describe a ◯ cutting the 1st ◯ at R Join SR With centre P and radius = the ⊥ from P on SR describe a *3rd* ◯ which will be touched by SR (III 16) From X draw XZO to touch the inner ◯ (III 17), and to cut the given ◯ (*i e 1st*) at Z, O, then OZ = SR, being chords at equal distances from the centre of the given ◯, (III 18 and III 14)

If X is **without** the ◯, Y must not be > the diameter If X is **within** the ◯, Y must not be > the diameter, and not > the chord drawn through X ⊥ to PX

431 If from an external pt two tangents are drawn to a ◯, the ∠ contained by them, is twice the ∠ contained by the chord of contact and the diameter drawn through one of the pts of contact

Let OX, OY be two tangents drawn from an *external* pt O to a ◯, whose centre is Z, and XZS the diameter through Z Then shall ∠ XOY be = 2 ∠ SXY Join ZO, cutting XY at R Then ZO bisects the ∠ XOY (III 17, Cor), and

it bisects XY at rt. ∠s, also XS is ⊥ to XO (III 18) Hence from the rt ∠d Δs—XOZ, RXZ, ∠XOZ = ∠RXZ or ∠YXS, each being the complement of the ∠XZO, ∴ ∠XOY = 2 ∠XOZ = 2 ∠YXS

432 Two ◯s touch one another *externally* and through the pt of contact, a st line is drawn terminated by the ◯ce, shew that the tangents at its extremities, are ||l

Let X be the pt of contact of the two ◯s, YXZ the st line through X terminated by the ◯ces, and YS, ZR the tangents at Y, Z *Then shall YS, ZR be* ||l Through X, draw SXR ⊥ to *the line of centre*, MN meeting YS, ZR at S and R Then SXR touches both ◯s at X (III 16) And ∵ SY = SX and RX = RZ (III 17, Cor), ∴ ∠ *SYX* = ∠ SXY = ∠ RXZ = ∠ *RZX*, *i e* ∠SYX = ∠RZX, ∴ YS ||l ZR (I 27)

433 If two ◯s which intersect, are cut by a st line, ||l to the common chord, show that the parts of it intercepted between the ◯ces, are equal

Let XY be the common chord of two ◯s, whose centres are S and R, and let DPQC ||l to XY cut one ◯ at D, C, and the other at P, Q Join SR, cutting DC at O Then SR is ⊥ to XY, ∴ SR is ⊥ to DC (I 29), OD = OC (III 3) and OP = OQ (III 3), ∴ OD—OP = OC—OQ Hence DP = CQ

434 Two ◯s touch one another internally Shew that, of all chords of the outer ◯, which touch the inner,—the greatest is that which is ⊥ to the st line joining the centres

Let X be the pt of contact, Y the centre of the *inner* ◯, Z the centre of the *outer* ◯. Then X, Y, Z are in the same st line (III 11) Let YZ produced, cut the inner ◯ce at P Let APB be the chord of the *outer* ⊙ which touches the *inner* ◯ at P, and is .. ⊥ to XP (III 18), draw DQ, any other chord touching the inner ◯ at M (III 17) From Z, draw ZE ⊥ to DQ, then E is *outside* the inner ◯ Let ZE cut the inner ◯ce at O Now ZO is > ZP (III. 7), and ∴ ZE > ZO (Ax 9), ∴ ZE > ZP, ∴ AB is > DQ (III 15)

435. Draw a ⊥ to a given st line, a st. line to touch a given ◯.

Let ZMN be the given ◯, and XY the given st line Draw the radius ZP ||l XY, and ZQ ⊥ to XY Then ZQ shall be a tangent to the ⊙ For ∠PZQ = ∠ZQX (I. 29) = a rt ∠ (constr); ∴ ZQ is a tangent to the ◯ (III. 16).

436 Two ○s have external contact at X, and a direct common tangent is drawn to touch them at B and C shew that a ○ described on BC as diameter is touched at X by the st line which joins the centres of the ○s

Let S, R be the centres of the two given ○s, then SR passes through X (III 12) At X draw the *common tangent* XY to meet BC at Y Then YB = YX and YC = YX (III 17, Cor) ∴ a ○ described on BC as diameter, passes through X, and touches SR, for XY is ⊥ to SR (III 16)

437 Two ○s intersect, and through one pt of section, any st line is drawn terminated by the ○ces, shew that the ∠ between the tangents at its extremities = the ∠ between the tangents at the pt of section

Let X be the pt of section of the two ○s, ZXO the st line through X terminated by the ○ces, and ZY, OY, the tangents at Z, O, and let the tangents at X meet ZY, OY at R, S Then shall ∠ ZYO = ∠ RXS Now, RZ = RX and SO = SX (III 17, Cor), ∴ △s RZX, SOX are isosceles, ∴ ∠ RZX = ∠ RXZ, and ∠SXO = ∠SOX (III 17, Cor), ∴ ∠ RZX + ∠ SOX = ∠YZO + ∠ YOZ = ∠RXZ + ∠SXO Hence ∠ZYO = ∠RXS (I 32, I 13)

438 Shew that two parallel tangents to a ○, intercept on any third tangent, a segment which subtends a rt ∠ at the centre

Let two ||l tangents XY, PQ touch the ○ at X, P, and cut off the segment YQ from a 3rd tangent whose pt of contact is Z Find O the centre (III 1) Join OY, OQ Then shall ∠QOY be a rt ∠ Join OP, OX, OZ Now OX, OP being ⊥ (III 18) to ||l st lines, are in the same st line Also QO, YO bisect the ∠s POZ and XOZ (III 17, Cor) Hence ∠ QOY = ½ of 2 rt ∠s (I 13) = one rt ∠

439 In a rt ∠d △, if a ○ be described from the mid point of the hypotenuse as centre, and with a radius = half the sum of the sides containing the rt ∠, it will touch the ○s described on these sides as diameters

Let XYZ be a rt ∠d △, and P the mid pt of XY the hypotenuse On YZ as diameter, describe a ○, let O be its centre Join PO and produce it to meet the ○ce at Q Then since PQ = PO + OQ, of which PO = ½ XZ (Ex 2-3, Text ps 96-97), and OQ = ½ YZ, *i e* PQ = ½ (YZ + ZX), hence if a ○ described with centre P and radius PQ, it will *touch* the ⊙ with YZ as diameter at Q,

since the centres of the two ⊙s and the pt. Q, are in the *same st line*

N B—The same ○ will touch the ○ on XZ.

440 If *three* ○s touch one another two and two, prove that the tangents drawn to them, at the *three* pts of contact, are *concurrent and equal*

Let X, Y, Z be the centres of the three ○s Then XY, YZ, XZ pass through B, C, A the pts of contact (III 12). Let the tangents at A and B meet at D Join CD Then CD shall touch the ○s whose centres are Y and Z, at C Join DX, DY, DZ Now since ∠s DBY and DAZ are rt ∠s (III 18), $DY^2=DB^2+BY^2$ and $DZ^2=DA^2+AZ^2$ (I 47) Since DA=DB (III 17, Cor), $DY^2-DZ^2=YB^2-ZA^2=YC^2-CZ^2$, ∴ CD is ⊥ to YZ, ∴ DC touches the ○s whose centres are Y and Z, DC=DA (III 17, Cor)=DB

441 If two ○s touch one another *internally*, and any third ○ be described touching both Then the sum of the distances of the centres of the third ○, from the centre of the two given ○s,—is constant

Let X, Z be the centres of the *outer* and *inner* ○s, and Y be the centre of any 3rd ○ touching the outer ○ at S and the inner ○ at O Then shall XY+YZ be constant Let P, R, Q be the radii of the 3 ○s Then the pts X, Y, S and Z, O, Y are *collinear* (III. 11 and 12) And XY + YZ=(P−Q) +(Q + R) =P+R

442 If the diameter XY of a ○, be produced to A, so that YA=the radius, through Y draw the tangent YPQ, and from A draw APO touching the ○ at O and meeting the former tangent at P, join XO and produce it to meet YPQ at Q, shew that the Δ QPO will be equilateral

Find S the centre (III 1) Join OS, OY Now ∠SOA=a rt ∠ (III 18), and Y the mid pt of SA (hyp), ∴ OY=YS=SO, and Δ SOY is equilateral And ∵ in the Δs XYQ, XYO, the rt ∠ XYQ=rt ∠XOY (III 31), and ∠X is common, hence ∠ OQP=∠OYS (I 32) Again the rt. ∠YOQ=rt ∠SOA (III. 18), rejecting the ∠ YOA which is common, ∠ QOP = ∠ YOS And ∵ in the Δs PQO, YSO, ∠ OQP= ∠OYS, and ∠ QOP= ∠ YOS, ∴ ∠OPQ=∠OSY (I 32) The ΔQOP being equiangular is equilateral (I 6, Cor).

443. If a ⊙ be described in one of the sides of a rt ∠d Δ, then the tangent drawn to it at the pt where it cuts the hypotenuse bisects the other side.

Let XYZ be the Δ, rt ∠d at X, and let the ○ on XY as diameter, meet YZ at Q. Then QP the tangent at Q, shall bisect XZ at P Join XQ Since ∠ YXZ is a rt ∠, XZ is tangent at X (III 16), ∴ PX=PQ (III 17, Cor), ∠PXQ=∠PQX And. ∠XQZ=a rt ∠, being supplement of ∠ XQY (a rt ∠) (III 31), ∴ ∠ PQZ= ∠ PZQ, ∴ PQ= PZ, ∴ PX=PZ

444 If one side of quadrilateral inscribed ○, be produced, the exterior ∠ is=to the opposite interior ∠ (See Notes on III 22)

445 Two *equal* segments of ○s are described on opposite sides of the same chord XY, and through A the mid pt of XY, any st line BAC is drawn, intersecting the arcs of the segments at B and C, shew that AB=AC

Let P and Q be the centres of two ○s, P, C and B, Q are on opposite sides of XY Join PQ, then PQ shall pass through A (Ex. 306) Join PC, BQ Then in Δs ABQ and PAC, PA= QA, PC=BQ, since the ○s are equal and ∠ PAC= ∠QAB (I 15), Δ PAC= Δ BAQ, for the ∠s PAC, BAQ are obtuse ∠s (*See Notes on I 26*), AB=AC

446 If two ○s intersect, and any no of st lines are drawn, one through each point of section terminated by the ○ces, shew that the chords which join them towards the same parts are ||l

Let the two ○s intersect at X, Y, and let AXB, CYD be two st lines terminated by the ○ces Join XY Now ∠ CAX+ ∠ CYX=2 rt ∠s (III 22) and ∠CYX= ∠ XBD (Ex 444), ∠ CAX+∠ XBD=2 rt ∠s, ∴ AC is ||l BD, (I 28)

447 XYZP is a quadril inscribed in a ○, and the opp sides XY, PZ are produced to meet at O, and ZY, PX, to meet at C, if the ○s circumscribed about the Δs OYZ, CYX intersect at S, shew that the pts O, S, C are *collinear*, or in one st line

Join OS, SC, SY Then OS, CS shall be in one st line For ∠ OSY = suppt of ∠OZY (III 22), which is=the suppt. of ∠ YXP (Ex 444)=suppt of ∠ YSC (Ex 444), ∴ O, S, C are collinear (I 14)

448 X, Y are any two pts in the ○ces of two segments, described on the same st line PQ and on the same side of it, the ∠s YPX, YQX are bisected by the st. lines PS, QS, meeting at S, shew that the ∠ PSQ is constant.

For ∠ PXQ is constant (III 21); ∠XPQ + ∠XQP is constant (I 32) So ∠YPQ + ∠YQP is constant. Hence their difference, the sum of the ∠s XPY, XQY is constant, and ∴ ∠SPY + ∠SQY is constant But it has been proved that ∠YPQ + ∠YQP is constant, ∴ ∠SPQ + ∠SQP is constant, and ∴ ∠PSQ is constant (I 32)

449 Find the *locus* of the centres of all ⊙s of given radius, which touch a given ⊙

Let Z be the centre of the given ⊙, ZX its radius, on ZX take XY = the given radius of the ⊙s, which are to touch the given ⊙ Now the ⊙ whose centre is Y and radius YX, will touch the given ⊙ at X. And ZY = ZX + YX or ZX ~ YX Hence the *required locus* is a ⊙, with centre Y and radius = the sum or difference of (the radius of the given ⊙ and the given radius of the touching ⊙s).

450 On YZ, ZX, XY the sides of a Δ XYZ, any no of pts A, B, C are taken, shew that the ⊙s described about the Δs XBC, YCA, ZAB meet in a pt

Let the ⊙s about the Δs YCA, ZAB intersect at O, join AO, CO, BO Then ∠ AOC = supplement of ∠ Y (III 22), and ∠AOB = supplement of ∠ Z (III 22) And since the 3 ∠ s AOC, AOB, BOC = 4 rt ∠s (I. 15, Cor) and ∠X + ∠Y + ∠Z = 2rt. ∠s (I 32), ∴ ∠COB = supplement of ∠ X, ∴ a ⊙ about Δ CXB will pass throught the pt O (III 22, *Converse*)

451. X, Y, Z are the mid pts of the sides PQ, QR, RP of a Δ PQR and A is the foot of the ⊥ PA let fall from one vertex P on the opposite side QR, shew that the points X, Y, Z, A are *concyclic*

Join YX, XZ, ZA, ZY Then ZA = ZR (Ex 107); ∴ ∠ZAR = ∠ZRA (I 6) = ∠ YXZ, for YRZX is a ▱gram (Ex. 2 p. 96, Text); ∴ ∠ZAY = ∠YXZ, ∴ A, Y, Z, X are concyclic. (III. 21, *Con*)

452 P, Q, R, S are four pts taken in order on the ⊙ce of a ⊙, the st lines PQ, SR produced intersect at X; and QR, PS produced meet at Y, shew that the st lines which respectively bisect the ∠ s PXR, PYR are ⊥ to each other

Let the st lines XO, YO bisect the ∠s PXR, PYR respectively Then YO, XO shall be ⊥ to each other For the ∠s PSR and PQR = 2 rt ∠s (III. 22), ∴ their supplements ∠ RSY + ∠ RQX = 2 rt ∠ s (I 13) But ∠XOY = ½ (∠ XQY + ∠ XSY), and ∴ = a rt ∠

453. Find the *locus* of the mid. pts. of chords of a ⊙ drawn through a fixed pt C.

Let O be the centre of the ⊙, and XCY any chord through C, meeting the ⊙ce at X and Y. Bisect XY at D (I. 10), ∴ D is the mid pt. of XY. Then ∠ CDO = a rt. ∠ (III. 3) and ∵ CO is a *fixed base*, the pt. D lies on the ⊙ce of a ⊙ on CO as diameter.

(1) When C is *outside* the ⊙, the *locus* is that part of the ⊙ on CO which is intercepted within the given ⊙.

(2) When C is *on* the ⊙ce, the *locus* is a ⊙ described on CO as diameter, and having internal contact with the given ⊙.

(3) When C is *inside*, the *locus* is a ⊙ within the given ⊙.

454. Find the *locus* of the pts. of contact of tangents drawn from a fixed pt. O to a system of concentric ⊙s.

Let C the centre of the concentric ⊙s, A be the point of contact of a tangent from O on any of these ⊙s. Then ∠OAC = a rt. ∠ (III. 18). And ∵ O and C are *fixed points*, the *locus* is a ⊙ on OC as diameter.

455. Two ⊙s intersect at P, Q, and through X any pt. on the ⊙ce of one of them, two st. lines XP, XQ are drawn, and produced if necessary, to cut the other ⊙ at A and B, find the *locus* of O, (the intersection of PB and QA).

In the △ PBO, the ext. ∠POQ = ∠*OQB* + ∠ OBQ (I. 32) = (∠ X + ∠ A) + ∠ B. ∵ ∠OQB = the ext. ∠ of △ XQA (I. 32), and X, A, B are all constant, being subtended by *fixed arcs*. ∴ ∠ POQ is constant, and ∵ P and Q are fixed, the *locus* of O is part of a ⊙ (III. 21, *Con.*)

When XP or XQ cuts the ⊙ce *without being produced*, the ∠POQ = supplement of ∠s (X + A + B). Thus the rest of ⊙ is obtained.

456. YXZ is any △ described on YZ the *fixed* base, and having a constant vertical ∠, and YX is produced to O, so that YO = to the sum of the sides containing the vertical ∠, find the *locus* of O.

Since OX = XZ (hyp.), ∴ ∠ XOZ = ∠XZO. But ∠ YXZ = ∠ XOZ + ∠XZO (I. 32). ∴ ∠ YXZ = 2 ∠XOZ or 2 ∠ YOZ, ∴ ∠ YOZ = ½ ∠ YXZ, and is ∴ constant, then since YZ is fixed, the locus of O is the arc of a segment YZ (III. 21, *Con.*)

457. Two ⊙s intersect at P and Q, and through P two st. lines XPY and APB are drawn terminated by the ⊙ces. If XA and BY intersect at O, shew that the points X, Q, B, O are *concyclic*.

Join XO, PQ, QY Then $\angle XQY + O = \angle PQY + \angle PQX + \angle O = \angle PQY + (\angle PAX + \angle O)$ (III. 21) $= \angle PQY + \angle PBY$ (I. 32) = two rt. $\angle$s (III. 22), $\therefore$ the pts X, Q, B, O are *concyclic* (III. 22, Con).

458 Given the base and vertical $\angle$ of a Δ, construct it so that its *area may be a maximum*

Since the base and vertical $\angle$ of a Δ are given, the vertex must be on the segment of a ⊙ described on PQ and containing the given $\angle$ (III. 21, *Con*), and the Δ of *greatest area* has the *greatest altitude* Now AB, (which bisects PQ at rt. $\angle$s at B), is the greatest altitude Take any other pt X on the arc of the segment Join BX and draw XO $\perp$ to PQ Now AB passing through the centre (III 1), is $>$ BX (III 7), and BX $>$ XO (I 18), $\therefore$ AB $>$ XO

459 Find a pt in a given st line PQ such that the tangents drawn from it to a given ⊙ contain the *greatest $\angle$ possible*

Let X be centre of the given ⊙, Draw XO $\perp$ to PQ; take M any pt in PQ The tangents from O shall contain a *greater* $\angle$ than the tangents from M From O and M draw tangents OS, MY Join XS XY, XM Then XM $>$ XO (I 18) Now $XM^2 = XO^2 + OM^2 = XY^2 + MY^2$, and $XO^2 = XS^2 + OS^2$, $\therefore (XS^2 + OS^2) + MO^2 = XY^2 + MY^2$, but $XS = XY$, $\therefore OS^2 + MO^2 = MY^2$, MY $>$ OS In YM make YZ = OS, join XZ. Then $\Delta XSO = \Delta XZY$ (I 4) $\therefore \angle SOX = \angle YZX$ But $\angle YZX > \angle YMX$ (I 16); $\therefore \angle SOX > \angle YMX$ But the $\angle$ between the tangents at O = 2 $\angle SOX$ and the $\angle$ between the tangents at M = 2 $\angle YMX$ (III 17, Cor), $\therefore$ tangents from O, include the *greater* $\angle$.

460. XY is the diameter of a ⊙, and Z is a given pt on the ⊙ce, such that the arc XZ is $<\frac{1}{2}$ the arc ZY, draw a chord OZ on one side of XY, so that arc OY may be thrice the arc XZ

Join Z with P the centre of the ⊙, and make the $\angle YPO = 3 \angle XPZ$ The arc YO is three times the arc XY (III 26)

461 Two ⊙s intersect at P and Q; and through P any st line XRY is drawn terminated by the ⊙ces Find the locus of E, the mid pt of XY

Let A and B be the centres of the two ⊙s Suppose E, (the middle pt of XY) fall *in* XP Bisect AB at C and draw AD, CF, BG $\perp$s. to XY Then $DG = \frac{1}{2}$ XY, for $DP = \frac{1}{2}$ XP, $PG = \frac{1}{2}$ PY (Cf Ex. 420); $\therefore$ DG = EY, each $= \frac{1}{2}$ XY and DF = FG (Ex 13, Text p. 98), DE = GY = PG, hence EF = FP. From the Δs CFE,

CFP, CE=CP (I 4), $\therefore$ the *locus* of *E* is a $\bigcirc$, with centre C and radius CP or CE

462 Of two $\bigcirc$s which intersect at P, Q, the $\bigcirc$ce of one passes through O the centre of the *2nd* From P any st line PXY is drawn (between Q and O), to cut them both at X and Y, shew that XQ =XY

Join PQ, OQ, OY, XQ and XO Then $\angle$ XOQ=$\angle$ XPQ (III 21) And $\angle$ YOQ=2 $\angle$ YPQ or 2 $\angle$ XPQ (III 20), $\therefore$ $\angle$ YOQ=2 $\angle$ XOQ, $\therefore$ $\angle$ YOX=$\angle$ XOQ Hence $\triangle$ YXO=$\triangle$ XOQ (I 4), $\therefore$ XQ=XY

463 XY is a fixed chord in a $\bigcirc$ and XA and BY any two ∥l chords through C and D, shew that AB touches *a fixed concentric* $\bigcirc$

Join XB, YA Now $\angle$ XYA= $\angle$ *XYB* (I 29) = $\angle$ *XAB* (III 21) And $\angle$ YXB=$\angle$ YAB (III 21), $\therefore$ the whole $\angle$AXB =the whole $\angle$ XAY But $\angle$XAY is constant, since XY is a fixed chord, $\therefore$ $\angle$ AXB is constant, $\therefore$ the arc AYB is fixed (III 26), $\therefore$ the chord AB is constant (III 29), $\therefore$ AB touches a fixed concentric $\bigcirc$

464 Given the base of a $\triangle$ and the sum of the remaining sides, find the *locus* of the foot of the $\perp$ from one extremity of the base on the bisector of the exterior vertical $\angle$

Let XYZ be one of a $\triangle$ on the fixed base XY Produce XZ to P and make ZP=ZY Now $\because$ XP=XZ+ZY, $\therefore$ XP is constant (hyp) Join YP, bisect $\angle$YZP by ZQ, meeting YP at Q Now ZQ bisects YP at rt $\angle$s (I 4) It is required to find the *locus* of Q Bisect XY at O and join OQ Now OQ=½ XP, (Ex 3, p 97 Text) $\therefore$ OQ is constant, and $\therefore$ O is a fixed pt, the *locus* of Q is a $\bigcirc$, with centre O and radius=½ XP

465 O is any point on the $\bigcirc$ce of a $\bigcirc$ with centre P circumscribed about a $\triangle$ XYZ, and $\perp$s OA, OB are drawn from O to the sides YZ, ZX Find the *locus* of the centre of the $\bigcirc$ circumscribed about the $\triangle$ OAB

Since $\angle$s OAZ and ZBO are rt $\angle$s, pts O, B, A, Z are concyclic (III 21, Con), $\therefore$ the $\bigcirc$ about $\triangle$ OBA passes through Z, and OZ is the diameter of the $\bigcirc$ about $\triangle$ OBA Bisect OZ at Q (I 10) It is reqd to find the *locus* of Q Join PQ Now $\therefore$ $\angle$ ZQP is a rt $\angle$ (III 3) and the pts Z, Q are fixed, $\therefore$ the *locus* Q is a $\bigcirc$ on the diameter PZ

466. Given PQ the sum of difference between two st lines, and the difference of their squares $=XY^2$, find the st lines

Draw YZ $\perp$ to XY (I 11) and of any length, and join XZ Then $XY^2=XZ^2-ZY^2$ (I 47) With centre P and radius $=XZ$ describe a ◯ With centre Q and radius $=YZ$ describe a ◯, cutting the 1st ◯ at O From O draw OD $\perp$ to PQ or PQ produced Then PD and QD shall be the st lines req For (1) $PD^2+DO^2=OP^2$ (I 47), and (2) $QD^2+DO^2=OQ^2$ (I 47), $\therefore$ taking (2) from (1), $PD^2-QD^2=PO^2-QO^2=XZ^2-ZY^2=XY^2$

467 Two chords PQ, XY of a ◯, whose centre is B, intersect at rt $\angle$s at A · shew that (a) $AP^2+AQ^2+AX^2+AY^2=4\,(\text{radius})^2$; (b) $PQ^2+XY^2+4AB^2=8\,(\text{radius})^2$

(*a*) Join XQ, PY Then $(AQ^2+AX^2)+(AY^2+AP^2)=XQ^2+PY^2$ (I 47) But since PQ, XY are at rt $\angle$s; $\therefore$ the arcs XQ+PY=a semi-◯ (Ex 1, p 222 Text) Hence $QX^2+PY^2=(\text{diameter})^2$, (III 31 and I 47) $=(2\,\text{radius})^2=(4\,\text{radius})^2$

(*b*) $PQ^2+XY^2+4AB^2=(AP^2+AQ^2+2AP\ AQ)+(AX^2+AY^2+2\ AX.\ AY)+4\,AB^2$, (II 4) $=4\,(\text{radius})^2+4\,AB^2+2\,AP\ AQ+2\,AX\ AY$ (from *Case I*) $=4\,(\text{radius})^2+2\,(AB^2+AP\ AQ)+2\,(AB^2+AX\ AY)=4\,(\text{radius})^2+2\,(\text{radius})^2+2\,(\text{radius})^2$, (III 35) $=8\,(\text{radius})^2$

468 Two tangents XP, XQ to a ◯, cut out one another at rt $\angle$s at X, find the pt on the intercepted arc PQ, such that the sum of the $\perp$s drawn from it to the tangents, may be a *minimum*

Let A be the middle pt of the arc PQ (III 30), and AB, AC the $\perp$s on XP, XQ Then AB + AC shall be a *minimum* Let Y be any other point on the arc PQ, and YZ, YS be the $\perp$s on XP, XQ, let AC, YZ intersect at D Join AY It is evident that, tangent at A makes equal $\angle$s with XP, XQ, and equal $\angle$s with AC, YZ If this tangent cuts YZ at L, L must be *outside* the ◯, and $\angle$DLA$=\angle$DAL But $\angle$DLA$>\angle$DYA (I 16), $\therefore$ $\angle$DAL$>\angle$DYA Much more $\angle$DAY$>\angle$DYA (Ax 9); DY$>$DA (I 19), and since DZ=AB and YS=DC, $\therefore$ (DY+DZ)+YS$>$(DA+DC)+AB, or YZ + YS$>$AC+AB or AC+AB$<$YZ+YS

469 Divide a given st line into two parts, so that the sum of the squares on the segments may be $=$ to (1) a given square, (2) may be *minimum*

Let XY be the given st line, and P the side of the given sq (1) At Y, make the $\angle XYZ=\frac{1}{2}$ a rt $\angle$ With centre X and the radius $=$ P, describe a ◯ cutting YZ at Q Draw QO $\perp$ to XY (I 12) Then XO^2+OY^2 shall be $=P^2$ For $\angle XYZ=\frac{1}{2}$ a rt $\angle$, and $\angle$ QOY$=$

a rt $\angle$, $\therefore$ $\angle OQY = \frac{1}{2}$ a rt $\angle$, $\therefore$ OY=OQ (I 6) Hence $XO^2 + OY^2 = XO^2 + OQ^2 = XQ^2$ (I. 47)=P^2 (hyp).

(2) Now $XO^2 + OY^2$ is a *minimum*, when XQ is a *minimum*, *i e* when XQ is $\perp$ to YZ Here $\angle QXY = \angle XYQ = \frac{1}{2}$ a rt $\angle$, $\therefore$ QX=QY, and $\therefore$ O is the middle point of XY

470 *Find a pt O on the* ⊙*ce of a* ⊙, *at which the st line joining two given pts, of which both are without the* ⊙ *subtends the greatest* $\angle$

Let X, Z be the given pts (both *external* to the ⊙) Through X, Z describe a ⊙ to touch the given ⊙ at Y Then $\angle XYZ$ shall be the *greatest* $\angle$ Let S be any other point on the ⊙ce of the given ⊙ Join XS, ZS and let XS meet the constructed ⊙ at O Join OZ Then $\angle XOZ > \angle XSZ$ (I 16) And $\angle XOZ = \angle XYZ = \angle XYZ$ (III 21), $\therefore$ $\angle XYZ > \angle XSZ$

471 *Describe the* Δ *of maximum area, having its* $\angle$*s = to those of a given* Δ *MNP, and its sides passing through three given pts A, B and C*

Join AB, AC, and on AB, AC describe segments containing $\angle$s N, P respectively Through A draw XY (the *maximum line*) terminated by the two ⊙ces Join XC, YB and produce XC and YB to meet at Z, then since $\angle X = \angle N$ and $\angle Y = \angle P$ (hyp), $\therefore$ $\angle Z = \angle M$ (I 32), and since $\angle$s of ΔXYZ are *fixed*, the *area is a maximum*, when any one of its sides is a maximum, but XY is a maximum, $\therefore$ Δ ABC has the *greatest area*

472 *Two* ⊙*s whose centres are P and Q intersect at X and Y, and a st line AXB is drawn through X and terminated by the* ⊙*ces Prove that (a)* $\angle AYB = \angle PXQ$, *(b)* $\angle AYP = \angle BYQ$

(1) From Δs YPQ and XPQ, we have $\angle XPQ = \angle YPQ$ (I 8); $\therefore$ $\angle XPQ = \frac{1}{2}$ $\angle XPY = \angle XBY$, (III 20) So $\angle XQP = \frac{1}{2}$ $\angle XQB = \angle XBY$ (III 20) from Δs PXQ and AYB $\angle PXQ = \angle AYB$, (I 32), (*b*) As in case first, $\angle PYQ = \angle AYB$ Take $\angle AYQ$ from each, $\therefore$ $\angle AYP = \angle QYB$

473 *Two equal* ⊙*s intersect at P and Q, and from O any point on the* ⊙*ce of one of them, a* $\perp$ *is drawn to PQ, meeting the other* ⊙ *at X and Y, shew that X is the orthocentre of the* Δ *OPQ*

Let the $\perp$ OC meet the first ⊙ at D and the second ⊙ at X and Y $\angle PDQ$ is the supplement of the $\angle$ POD (III 22) And $\therefore$ segment PDQ=segment PXQ (Hyp and III 28), $\angle PDQ = \angle PXQ$, $\therefore$ $\angle PXQ$ is supplement of $\angle POQ$, $\therefore$ the

orthocentre of $\triangle$ OPQ is on arc PXQ, (Ex. 21, p. 226 Text). But the orthocentre is on $\perp$ OC, $\therefore$ *orthocentre is at X.*

474. *From a given pt. Y, without a ⊙, draw a st. line to the concave ⊙ce, so as to be bisected by the convex ⊙ce.*

Let X the centre of the given ⊙. Join XY, and bisect it at Q. With centre Q and radius = ½ radius of the given ⊙, describe a 2nd ⊙, cutting the 1st ⊙ at Z. Join YZ and produce it to meet the ⊙ce of the 1st ⊙ at P, then shall YZ be = ZP. $\because$ Q is the middle point of XY (Cons.) and QZ = ½ PX, for QZ is the radius of the 2nd ⊙ = ½ the radius of the given ⊙, $\therefore$ YZ = ZP.

475. *Two ⊙s cut one another* orthogonally *at X and Y; M is any point on the arc of one ⊙ intercepted by the other, and XM, YM are produced to meet the ⊙ce of the second ⊙ at N and P. shew that NP is a diameter.*

Let A, B be the centres of the two ⊙s, M being on the ⊙ce of ⊙ whose centre is A. Join AX, XY, BX, XP. Then AX and BX are tangents, $\therefore$ ∠AXN = ∠XPN (III. 32) and ∠BXM or ∠BXN = ∠XYM (III. 32) = ∠XNP (III. 21) = ∠AXB = ∠XPN + ∠XNP, but ∠AXB = a rt. ∠, (⊙s cut *orthogonally*), ∠XPN + ∠XNP = a rt. ∠ rem. ∠PXN = a rt. ∠; $\therefore$ NP is a diameter (III. 31).

476. *XYZ is a $\triangle$ inscribed in a ⊙, and P, Q, O, are the mid. pts. of the arcs subtended by the sides remote from the opposite vertices, find the relation between the ∠s of the two $\triangle$s XYZ, PQO and prove that $\triangle$ ABC (the* pedal *$\triangle$ of PQO), is equiangular to $\triangle$ XYZ.*

Join XO. Since the arc XP = ½ arc XZ, and arc XQ = ½ arc XY, $\therefore$ ∠XOP = ½ ∠XOZ = ½ ∠XYZ, and ∠XOQ = ½ ∠XOY = ½ ∠XZY, (III. 27 and 21). Hence ∠XOP + ∠XOQ or ∠POQ = ½ (Y + Z). So ∠QPO = ½ (X + Z), and ∠PQO = ½ (X + Y).

Again XO makes with PQ an ∠ = that at the ⊙ce, subtended by the sum of the arcs XP, YO + YQ (Ex. 323); *i.e.* an ∠ = ½ (X + Y + Z) = one rt. ∠ (I. 32). Hence XO, YP, ZQ are ⊥s of sides $\triangle$ POQ. Then $\because$ XO bisects ∠BAC, (Ex. 20, p. 225 Text), $\therefore$ ∠QAB = ∠PAC = ∠QOP (Ex. 20, p. 225 Cor. 2) = ½ ∠(Y + Z), $\therefore$ ∠BAC = 2 rt. ∠s − ∠QAB − ∠PAC = 2 rt. ∠s − (Y + Z) = ∠X. So ∠ACB = ∠XZY and ∠ABC = ∠XYZ, $\therefore$ ΔABC is equiangular to Δ XYZ.

477. *Two tangents AX, AY are drawn from an external pt. A to a given ⊙, and Z is the mid. pt. of the chord of contact XY, if PQ be any chord through A, shew that XY bisects ∠PZQ.*

Let C be the centre of the given ⊙. Join AZ, then AZ produced must pass through C. Join CQ, CX, CP. Since the $\angle$s AXC, XZC are rt. $\angle$s, AX must touch the ⊙ described about $\triangle$ XZC. ∴ $AZ.AC = AX^2 = AP.AQ$ (III. 36), ∴ the four pts P, Z, C, Q are *concyclic*, (III. 35, *Con.*), ∴ $\angle PZA = \angle CQP$ (Ex. 444) $= \angle CPQ$ (I. 5) $= \angle CZQ$ (III. 21). And XZ is ⊥ to AC, ∴ ZX or XY bisects $\angle PZQ$.

478. *Given the sum of two st. lines, and the sum of the squares on them, find the st. lines.*

Let X be the sum of the req. st. lines, and let AB^2 = the *sum* of the sqs. on them. On AB describe a semi-⊙ and a segment containing an $\angle = \frac{1}{2}$ a rt. $\angle$ (III. 34). With centre A, and radius X, draw a ⊙ cutting the 2nd segment at P. Join AP cutting the semi-⊙ at Q. Join QB and PB. Then shall AQ, QB be the st. lines reqd. Now since $\angle AQB$ is an $\angle$ in a semi-⊙, ∴ $\angle AQB$ = a rt. $\angle$ (III. 31), ∴ $AQ^2 + QB^2 = AB^2$ (I. 47). And exterior $\angle AQB = \angle APB + \angle QBP$ (I. 32). But $\angle APB = \frac{1}{2}\angle AQB$ (cons.), ∴ $\angle QBP = \frac{1}{2}\angle AQB$, ∴ $\angle APB$ or $\angle QPB = \angle QBP$, ∴ $QP = QB$, ∴ $AQ + QB = AQ + QP = AP = X$.

479. *XYZ is a $\triangle$, and the internal and external bisectors of the $\angle X$, meet YZ at Q, and YZ produced at P; if A be the mid point of QP, shew that AX is a tangent to the ⊙ described about the $\triangle$ XYZ.*

Now $\angle QXP$ = a rt. $\angle$ (Ex. 2 p. 29, Text), ∴ $AQ = AX = AP$ (Ex. 107). And $\angle AXQ = \angle AQX$ (I. 5). Now $\angle AXZ = \angle AXQ - \angle ZXQ = \angle AQX - \angle QXY$ (hyp.) $= \angle XYZ$ (I. 32), ∴ AX is a tangent to the ⊙ about the $\triangle$ XYZ (III. 31, *Con.*)

480. *The st. lines XO, OY, ZO which join the vertices X, Y, Z of a $\triangle$ XYZ, to the centre O of its circumscribed ⊙, are ⊥ respectively to the sides of the* pedal *$\triangle$ PQR.*

Let OX meet PQ at C. Now $\angle XPC = \angle XYZ$, (See Ex. 20, Cor. p. 225 Text), and $\angle XOY = 2\angle XZY$ (III. 20), ∴ $\angle OXY$ = complement of $\angle XZY$ (I. 5, I. 32), ∴ from $\triangle$ XPC, $\angle XCP$ is a rt. $\angle$ (I. 32).

481. *X is any pt. on the ⊙ce of a ⊙ circumscribed about a $\triangle$ PQR; shew that the $\angle$ between* Simson's Line *DBA for the point X, and the side QR = the $\angle$ between PX and the diameter of the circumscribed ⊙.*

Let XD be a ⊥ on PR produced, XB, XA ⊥s on QR and PR. Draw the diameter PC and join PX and XC. Then pts X, B, A, R are *concyclic*, ∴ $\angle ABR = \angle AXR$ (III. 21) = complement of $\angle XRP$ = complement of $\angle XCP$ (III. 31) $= \angle XPC$ (III. 31).

*482 *XYZ is a △, and from any point O within it, ⊥s OP, OQ OR are drawn to the sides, if A, B, C are the centres of the ◯s circumscribed about the △s QOR, OPQ, POR, shew that (1) the △ ABC is equiangular to the △ XYZ, and (2) that the sides of the one, are respectively = ½ of the sides of the other.*

Join OX, OY, OZ, and bisect them at A, B, C. Then since ∠s OQX and ORX are rt ∠s, the four pts O, R, X, Q lie on a ◯, whose diameter is OX (III 22, Con), ∴ A is the centre of the ◯ about the △ QOR. So for B, and C Also BA, AC, BC are ∥ to XY, XZ, YZ and = half of these lines (Ex. 2-3, ps 96-97 Text); ∴ ABC is equiangular to the △ XYZ

483 *Three ◯s intersect at P, and their other pts of intersection are Q, R, S, QP cuts the ◯ RPS at X, and XR, XS cut the ◯s QPR, QPS respectively at Y and Z, shew that the points Y, Q, Z are* collinear

Join QY, QZ. Then shall QY QZ be in the same st line Join PR, PS (1) Since the four pts Z, Q, P, S are *concyclic*, ∠ZQP = ∠XSP (Ex 444) And ∵ the *four* pts Y, Q, P, R are *concyclic*, ∴ ∠YQP = ∠XRP (Ex 444), the ∠s ZQP, YQP = ∠s XSP, XRP = two rt ∠s, for the pts X, R, P, S are concyclic (III 22), ∴ ZQ, QY are in the same st line.

484 *A and B are two fixed pts in the diameter of a ◯ equidistant from the centre O, through A, any chord DAE is drawn, and its extremities are joined to B, shew that the sum of the square on the sides of the △ DEB, is constant*

Let PQ be the diameter Now $DE^2 = DA^2 + EA^2 + 2DA \cdot AE$ (II 4), adding $DB^2 + EB^2$ we have, $DE^2 + DB^2 + EB^2 = (DA^2 + DB^2) + (EA^2 + EB^2) + 2\, DA \cdot AE$ But $DA^2 + DB^2 = 2DO^2 + 2AO^2$, and $EA^2 + EB^2 = 2EO^2 + 2AO^2$ (Ex 245), which are constant And $DA \cdot AE = PA \cdot AQ$ (III 35), which is constant, ∴ $DE^2 + DB^2 + EO^2$ is constant

485 P *is a fixed pt and XY is a fixed st. line of indefinite length, PQ is any st line drawn through P to meet XY at Q, and in PQ, a point Z is taken such that, the rect PQ. PZ is constant; find the* locus *of Z*

Draw PO ⊥ to XY (I. 12), and from Z draw ZS at rt. ∠s to PQ meeting PO at S (I 11) Then ∵ ∠s QZS and SOQ are rt ∠s, ∴ the pts Q, O, S, Z are *concyclic* (III. 22, Con), ∴ PS PO = PZ PQ (III. 36), since PQ PZ is constant (hyp), ∴ PO PS is constant, and since PO is constant, ∴ PS is constant; ∴ S is a fixed pt. And ∠ PZS is a rt. ∠, ∴ the *locus* of Z, is a ◯, on PS as diameter.

486 Two ⊙s intersect at P and Q, and from any point Q on the ⊙ce of one of them XP, XQ are drawn and produced if necessary to meet the other at Y and O, shew that YO is ∥l to the tangent at X

Let XZ be the tangent at X, then ∠ZXP = ∠XQP (III 32) = ∠PYO (Ex 444), ∴ ZX is ∥l YO (I 27)

487 XYZ and PQR are two △s inscribed in a ⊙, so that XY, XZ are respectively ∥l to PQ, PR, shew that YR is ∥l to QZ

Since PQ, PR are respectively ∥l to XY, XZ, ∴ ∠P = ∠X, ∴ arc QR = arc YZ (III 26) From these equal arcs, take the arc YR, then arc YQ = arc ZR, ∴ ∠QZY = ∠ZYR (III 27), ∴ QZ is ∥l YR (I 27)

488 A secant ABC and a tangent AO are drawn to a ⊙ from an external pt A, and the bisector of the ∠BOC meets BC at P, shew that AP = AO

For ∠AOB = ∠OCB (III 32), and ∠BOP = ∠POE Hence ∠AOP = ∠AOB + ∠BOP = ∠PCO + ∠POC = ∠OPB (I 32), ∴ AO = AP (I 6)

489 Two segments of ⊙s are on the same chord PQ and on the same side of it, and X and Y are any pts one on each arc, find the *locus* of the intersection of the bisectors of the ∠s XPY and XQY

Let the bisectors meet at O The *locus* of O is required Now ∠OPQ = ½ ∠s (XPQ + YPQ), and ∠OQP = ½ ∠s (XQP + YQP), ∴ the sum of the ∠s at the base of △ OPQ = *one-half* of the sum of the ∠s at the base of △s XPQ, YPQ Hence the vertical ∠POQ = *one-half* of vertical ∠s PXQ, PYQ (I 32), both of which are constant (III 21), ∴ ∠POQ is constant, and P, Q are fixed pts, ∴ the *locus* of O is the arc of a segment of ⊙ on base PQ (III 21, Con)

490 A is any pt within a △ XYZ, and AB, AC, AD are drawn ⊥s to YZ, ZX, XY respectively, shew that ∠YAZ is = ∠YXZ + ∠DBC

Join XB The points D, Y, B, A are *concyclic*, ∴ ∠YAB = ∠YDB (III 21) = ∠DXB + ∠DBX (I 32) So ∠ZAB = ∠ZCB = ∠BXC + ∠CBX Hence by *addition*, ∠YAZ = ∠YXZ + ∠DBC

491 Two ⊙s intersect, and through a pt of section a st line is drawn bisecting the ∠ between the diameters through that pt, shew that this st line cuts off similar segments from the two ⊙s

Let P be the pt. of intersection of the two ◯s, PQ, PR the two diameters; and let the st line through P, meet the ◯s at A and B. Then in the △s APQ, PBR; ∠ QPA = ∠RPB, and ∠ PAQ = a rt ∠ PBR (III. 31), ∴ ∠ PQA = ∠BRP (I 32) ∴ the segments are similar.

492 A is any pt on the ◯, of which PQ is a *fixed diameter*, and AB is drawn ⊥ to PQ. On PB and QB as diameters, ◯s are described, which are cut by PA, QA at R and S, shew that RS is a *common tangent* to these ◯s.

Join RB, SB. Then each of the ∠s PRB, BSQ, PAQ is a rt ∠ (III 31), ∴ the points R, B, S, A are *concyclic*, ∴ ∠BRS = ∠ BAS = ∠ BPR from the rt ∠ d △s APQ, BAQ (I 32), ∴ RS touches ◯ PRB (*Converse of III 32*)

493 The ⊥s drawn from the vertices of a △, to the opposite sides are concurrent (*See Text p 224, Ex 19*)

494 PQ is a fixed diameter of a ◯; and XY a fixed st line of indefinite length cutting PQ or PQ produced at rt ∠s, any st line is drawn through P to cut XY at O, and the ◯ at Z, shew that (1) the rectangle PO PZ is constant, and (2) PO PZ = PX^2.

Let PQ and XY intersect at S. Join ZQ, XQ. Then ∠s OZQ, OSQ are rt ∠s (III 31 and hyp), ∴ the pts Z, O, Q, S are *concyclic*, ∴ PO PZ = PS PQ (III 36, Cor) And ∵ PS and PQ are constant, ∴ rect PZ PO is *constant*.

(2) Again the ◯ about the △ XSQ has its centre on XQ; for ∠ XSQ is a rt ∠ (III 31), and PX is ⊥ QX (III 31). ∴ PX is tangent to the ◯ about △ QSX ∴ PS PQ = PX^2 (III 36); ∴ PO PZ = PX^2

495 Given the base and vertical ∠ of a △, find the *locus* of its *orthocentre* (*See Text p 227, Ex 35*)

496 Through the extremities of a given st line PQ, any two ∥ *st lines PX, QY are drawn, find the* locus *of the intersection of the bisectors of the* ∠*s XPQ, YQP*

Let the bisectors meet at O. Then ∠XPQ + ∠YQP = 2 rt. ∠s.

496 Through the extremities of a given st line PQ, any two ∥ *st lines PX, QY are drawn, find the* locus *of the intersection of the bisectors of the* ∠*s XPQ, YQP.*

Let the bisectors meet at O. Then ∠ XPQ + ∠YQP = 2rt ∠s (I. 29); ∴ ∠ OPQ + ∠OQP = one rt ∠(hyp); ∴ ∠ POQ = one rt. ∠ (I. 32). And since PQ is *fixed*, the *locus* of O is a ◯, on PQ as diameter (III 31, *Con*)

497 Describe an equilateral Δ, so that its sides may pass through *three* given points

Let A, B, C be the *fixed* pts On AC, AB describe (*externally* to the Δ ABC), segments containing an ∠ of an equilateral Δ (III 33) Through A, draw any st line XAY terminated by the ○ces Join XB, YC, and produce them to meet at Z Then ΔXYZ is an equilateral Δ Since each of the ∠s X, Y is ⅓ of two rt ∠s, the ∠ Z is = ⅓ of two rt ∠s

498 XYZ is a Δ inscribed in a ○, and from any point A on the ○ce, AP, AQ are drawn ⊥s to XY, YZ, if PQ, or PQ produced, cuts XZ at O, shew that AO is ⊥ to XZ

Since ∠APY = a rt ∠ = ∠ AQY, pts A, P, Y, Q are *concyclic* (III 22, *Con*), ∠ AQP = ∠ AYP (III 21) = ∠ XZA (Ex 444) or ∠ OZA, ∴ ∠AQP + ∠AQO = ∠OZA + ∠AQO, but ∠AQP + ∠AQO = two rt ∠s (I 13), ∴ ∠OZA + ∠AQO = two rt. ∠s, ∴ the pts A, Q, O, Z are *concyclic* (III 22, *Con*), ∠AOZ = ∠ AQZ (III 21) = a rt ∠ (cons)

499 Describe a ○ to pass through two given pts, and to touch a given ○ (*See Text p 236, Ex 22*)

500 A is the centre of a ○, and AP, AQ two fixed radii, if from any point O on the arc PQ ⊥s OX, OY are drawn to AP, AQ, shew that XY is *constant*

Since ∠s AXO, AYO are rt ∠s ∴ the *four* pts A, X, O, Y are *concyclic* (III. 22, Con), ∴ AO is a diameter of ○AXOY (III 31) Also ○ AXOY is constant, for AO is a radius of the given ○ Now ∠ XAY is constant, ∴ the chord XY is constant

501 On the three sides of any Δ, equilateral Δs are described *remote from the given* Δ; shew that the ○s described about them, intersect at a point

Let PQR be the given Δ, on the sides of which, equilateral Δs are described *externally*, and let the ○s about the equilateral Δs on QR, RP, meet at A Join PA, QA, RA Since the ∠ of an equilateral Δ = ⅓ of two rt ∠s, ∴ each of the ∠s PAR, QAR is ⅔ of two rt. ∠s (III 22), ∴ ∠ PAQ is = ⅔ of two rt ∠s (I 15, Cor), ∴ a ○ described about the equilateral Δ on PQ, will pass through A (III 22, *Con*)

502 Find the *locus* of a pt A which moves, so that if ⊥s AD, AB, AC, are drawn from it to the sides XY, YZ, ZX of a given Δ XYZ, their feet D, B, C, are *collinear*.

Since the pts. A, D, Y, B are *concyclic* (III. 22, *Con.*), ∴ ∠BAY = ∠BDY or ∠CDY (III. 21). And since the pts. A, Z, C, B are *concyclic* (III. 22, *Con.*), ∴ ∠*BAZ* = supplement of ∠BCZ (III. 22) = ∠*BCX*, the whole ∠YAZ = ∠CDX + ∠BCX = supplement of ∠X (I. 32), ∴ A lies on the ⊙ce of the ⊙ described about △XYZ.

503. PQR is a △, rt. ∠d at R, and from X, any point in the *hypotenuse* PQ, a st. line XY is drawn ⊥ to PQ, and meeting QR, at Y, shew that $XY^2 = PX.XQ - RY.YQ$.

Since ∠s PXY, YRP are rt. ∠s, ∴ the pts. X, P, R, Y are *concyclic* (III. 22, *Con.*), ∴ QY.QR = QX.QP (III. 36, Cor.), or $QY.YR + QY^2 = QX.XP + QX^2$ (II. 3), or $QY^2 - QX^2 = QX.XP - QY.YR$; ∴ $XY^2 = QX.XP - QY.YR$ (I. 47).

504. Describe a ⊙ to touch two given st. lines, and a given ⊙. *(See Text p. 237, Ex. 24.)*

505. A semi-⊙ is described on PQ as diameter, and any two chords PX, QY are drawn intersecting at Z: shew that $PQ^2 = PX.PZ + QY.QZ$.

Draw ZO ⊥ to PQ. Join PY, QX. ∵ ∠s ZOQ, ZXQ are rt. ∠s (Cons. and III. 31), the pts. Z, O, Q, X are *concyclic*, (III. 22, *Con.*), PZ.PX = PO.PQ (III. 36, Cor.). So QZ.QY = QO.QP (III. 36, Cor.), ∴ PZ.PX + QZ.QY = PO.PQ + QO.QP = PQ^2 (II. 2).

506. If from a point X, without *a ⊙, XY is drawn ⊥ to a diameter PQ, and a* secant *XAB, shew that $XY^2 = XA.XB + PY.YQ$.*

Let XY meet the ⊙ce of the given ⊙ at R, S. Then RS is bisected at Y, and produced to X, $XY^2 = XR.XS + YR^2$ (III. 6) = *XA.XB + PY.YQ*, for XR.XS = XA.XB (III. 36, Cor.) and $YR.YS = YR^2 = PY.YQ$ (III. 35).

507. Two st. lines XY, XZ of indefinite length touch a given ⊙ at P and Q, and any chord BO is drawn, so as to be bisected by the *chord of contact* PQ, if BO is produced, shew that the *intercepts* between the ⊙ce and the tangents (BS, OR) are equal.

Let the chord BO be bisected by PQ at C, and produced to meet XY and XZ at S and R. Then shall BS = OR. Find the centre M (III. 1). Join MR, MQ, MC, MS, MP. Now ∠MCR = a rt. ∠ (III. 3), ∠MQR = a rt. ∠ (III. 18), ∴ the *four* pts. M, C, Q, R are *concyclic* (III. 21, *Con.*), ∴ ∠CRM = ∠CQM (III. 21). So ∠CSM = ∠CPM (III. 21). But MQ

$=MP$, $\therefore \angle CQM = \angle CPM$ (I 5), $\angle CSM = \angle CRM$. $\because CR=CS$ (I 26) And $CB=CO$ (hyp), $BS=OR$

508 Describe a ○ to pass through two given pts Y and Z, and to intercept an arc of given length, on a given ○ PQA

Find C the centre of the given ○ PQA (III 1) Join PQ, and with centre C, describe a ○ to touch the chord PQ The given ○ PQA, intercepts on every tangent to the constructed ○, a part $=PQ$ Draw a ○ YBZ through Y, Z touching the given ○ at B (Ex 499) Draw XB (the common tangent of ○s PQA, YBZ), to meet ZY produced at X From X draw a tangent to the constructed ○ meeting the given ○ PQA at O, A Now $XY \cdot YZ = XB^2 = XO \cdot XA$ (III 36), $\therefore$ Y, Z, O, A are *concyclic* (III 35, *Con*), $\therefore$ a ○ described through the pts Y, Z, O will pass through A And $OA = PQ$, $\therefore$ arc $OA =$ arc PQ (III 28)

***509 If from any pt P on the ○ce of a ○, described about △ ABC, ⊥s are drawn to the three sides AB, BC, CA, the feet (F, D, E) of these ⊥s, are in the same st line** *(See Text p 282, Ex 74)*

510 AB is the diameter of a ○, and BP any chord, BQ bisects ∠ ABP Shew that BQ divides the area *ABP into two* unequal *parts* (C U F A Pap 1873)

Join AQ, PQ Since $\angle PBQ = \angle ABQ$ (hyp), $\therefore$ arc $PQ=$arc AQ (III 26), $\therefore$ chord $PQ=$chord AQ (III 29) Now in two △s ABQ, BPQ, since $AQ=PQ$ and BQ common, and $AB>BP$ (III 15), $\therefore \angle AQB > \angle PQB$ (I 25) At the pt Q in the st line BQ, make $\angle BQC = \angle PQB$ (I 23) Then $\triangle PBQ = \triangle QBC$ (I 26) Thus $\triangle AQB$ is $> \triangle BQC$ Hence the *area* of $\triangle ABQ$ is $>$ area of $\triangle BPQ$

511 *PQR is a △ inscribed in a ○, and the bisectors PA, QB, RC of its ∠s P, Q, R which intersect at X, are produced to meet the ○ce in A, B, C, shew that X is the* orthocentre *of the △ ABC*

Let PA meet CB in O In the △ ACO, $\angle OAC$ or $\angle XAC = \angle PRC$ (III. 21) $= \frac{1}{2} \angle R$, and $\angle ACO = \angle ACX + \angle XCO = \angle RPO + \angle XQR$ (III 21) $= \frac{1}{2}(P+Q)$, $\therefore \angle XAC + \angle ACO = \frac{1}{2} \angle (P+Q+R) =$ one rt $\angle$ (I 32), $\therefore$ PA is ⊥ to CB So CR is ⊥ AB, and BQ is ⊥ AC, $\therefore$ X is the *orthocentre* of △ABC

512 The ***greatest rectangle*** **which can be inscribed in a ○, is a** ***square***

Let XYZS be a rectangle, and XLZM the square inscribed in the ○ XYZ, having the same diagonal XZ Then shall XLZM

be $>$ XYZS Through M draw MP ||| XZ, produce XS to meet MP at P, join ZP, Δ XMZ = Δ XPZ (I 37), but Δ XPG $>$ ΔXSZ, $\therefore$ ΔXMZ $>$ Δ XSZ So, it may be shewn that Δ XLZ $>$ ΔXYZ Hence, square XLZM $>$ rectangle XYZS So, it can be proved that, square XLZM is $>$ than any other rectangle inscribed the ◯, . square is the greatest rectangle.

513 Given PQ the *base*, X the *altitude*, and O (the radius of the circumscribed ◯); construct the Δ

Bisect PQ at A (I 10) Draw AB at rt $\angle$s to PQ, making AB = X (the given *altitude*) Through B, draw MBN ||| to PQ (I 31) With centre P, and radius = O, describe a ◯ cutting AB at C It is evident from (III 1), that the centre of the ◯ *about* the Δ to be constructed, must be on AB With centre C, and radius = O, describe a ◯ cutting MN at X and Y Join PM, MQ, PY, YQ Then Δ PMQ or Δ PYQ is the Δ required

514 Given the base XY, the vertical $\angle$, and M^2 = the *sum* of the squares on the sides containing the vertical $\angle$, construct the Δ

Describe a segment containing the *given vertical* $\angle$, on XY (the given *base*) (III 33), the vertex of the reqd Δ must be on the *arc* of this segment. Bisect XY at P (I 10) It is required to find a pt. Q on arc XQY, such that $XQ^2 + YQ^2 = M^2$ But $XQ^2 + YQ^2 = 2(XP^2 + PQ^2)$ $2(XP^2 + PQ^2) = M^2$ But XP is known, and M is given, hence by (II 14 and I. 47), PQ can be found With centre P, and radius PQ, describe a ◯ intersecting arc XQY at Q, R Now, either Δ XYQ or Δ XYR is the Δ required

515 Given the vertical $\angle$, one of the sides containing it, and the length of the $\perp$ from the vertex on the base, construct the Δ

Let $\angle$ OXQ be the *given vertical* $\angle$, OQ the given side, and M = length of $\perp$ from the vertex on the base With centre O and radius = M, describe a ◯, and from Q, draw QYR to touch the ◯ at Y, and meet OX at R Then OQR is the reqd Δ For, OY is $\perp$ to QR (III. 18), and is = M *the length of the $\perp$ from the vertex on the base*

516. Describe a ⊙ to pass through two given pts and to touch a given st line. *(See Text p 235, Ex. 21)*

**517. Given the vertical $\angle$, and the lengths of two st lines drawn from the extremities of the base to the pts. of bisection of the sides; to construct the Δ*

Let AB = *one* of the *given* st. lines, bisect AB in C (I. 10), and

on AC describe a segment of a ◯ containing an $\angle$ = the given $\angle$ (III. 33), from BA cut off BD = ⅓ of BA, from D, draw DE to the ◡ making DE = ⅓ of the *other* given st line Produce ED to F, making DF = 2DE Join AE, FB, and produce AE, FB to meet in G. Join AF Now AFG is the Δ reqd Join CE Since BD = 2DC, and DF = 2DE, and EC is ||| to FG, ∴ $\angle$AGF = $\angle$AEC = the given $\angle$ Hence AE = EG, and BF being = 2 EC = BG, ∴ AB and ΓE, (*equal to the given st lines*) are drawn to the pts of bisection of the sides

518 Given PQ the base, the vertical $\angle$, and the difference of the squares on the sides containing the vertical $\angle$, construct the Δ

On PQ, describe a segment of a ◯, containing the given $\angle$ (III 33), the vertex of the reqd Δ must lie on the arc of this segment In PQ, take a pt X, so that $PX^2 - QX^2$ = the *given square* From X, draw XR $\perp$ to PQ, to meet the arc at R Then the Δ PQR is the reqd Δ Since $PR^2 = PX^2 + XR^2$, and $RQ^2 = XQ^2 + XR^2$, ∴ $PR^2 - QR^2 = PX^2 - QX^2$ = the *given square*

519 Describe a ◡ to pass through a given pt. and to touch two given st lines *(See Text p 236, Ex 23)*

520 If a st line be divided into any two parts, to produce it, so that the rectangle contained by the whole *line so produced, and the* part *produced, may be = to the rectangle contained by the given st line and one segment*

Let AB be the given st line, divided into two parts in the pt C On AB as diameter, describe a ◯ ADB From B, draw BE at rt $\angle$s to AB, and ∴ a tangent to the ◡ (III 16) Take BS, such that BE^2 = AB AC Find O the centre, join EO, and produce EO to F Produce AB to G, making BG = ED Then shall the rectangle AG GB be = the rectangle BA AC Since DE = BG, ∴ BG GA = DE EF, (III 36, Cor) = EB^2 (III 36) = AB AC (cons)

** 521 To divide a given st line into two such parts, that the rect contained by the* whole *line and* one *of the parts, may be* (m) *times the square on the other part*, (m *being whole or fractional*)

Let AB be the given st line, and in it produced, take BC = an *m*th part of AB On AC, describe a semi-◯, and from B draw BD $\perp$ to AC Bisect CB in O (I 10), join OD, and take OE = OD, then AB will be divided in E, as required On BC describe a semi-◯, cutting OD in F, join EF. Then $\angle$DOE being common to the Δ s DOB, EOF, and DO = OE, BO = OF, the Δs will be *equiangular* and *equal*, and ∴ $\angle$ OFE = $\angle$OBD, and ∴ a rt. $\angle$, hence FE is a *tangent* to the ◯ CFB (III. 16) Hence AB BC

$=DB^2=FE^2=CE\cdot EB$ (III. 36) From each of these equals, take away $CB\cdot BE$, and $AE\cdot CB=BE^2$, ∴ (m) times the rect $AE\cdot CB$ or the rect $AB\cdot AE=(m)$ times BE^2

**522 If from any point P on the ⌒ce of a ○, ⊥s be drawn to the four sides and to the diagonals of an inscribed quadrilateral, the rect contained by the ⊥s on either pair of opposite sides = rect. contained by the ⊥s on the diagonals*

Let PE, PF be the ⊥s on the opposite sides AB, CD, PG, PH the ⊥s on the diagonals Join EG, FH, PA, PD Since EAGP, FDHP are *cyclic* quadrilaterals, ∠ PEG = ∠ PAG or ∠ PAC, and ∠ PHF = ∠ PDF (III 21) Since PACD is a *cyclic* quadrilateral, ∠ PAC + ∠ PDC = 2 rt ∠s (III 22) = ∠ PDF − ∠ PDC, ∴ ∠ PAC = ∠ PDF (Ex 444), ∠ PEG = ∠ PHF, from above So ∠ EAP = ∠ EGP and ∠ PFH = ∠ PDH or ∠ PDB (III 21), but ∠ EAP = ∠ PDB or ∠ PDH (Ex 444), ∴ ∠ EGP = ∠ PFH Hence Δs PEG and PFG are *equiangular*, ∴ PE·PF = PG·PH (For, in equiangular Δs, the rectangle under the *non-corresponding sides* about equal ∠s, are = to one another) (See Ex 38)

523 Given the base, and vertical ∠ of a Δ, find the *locus* of the intersection of the bisectors of its ∠s *(See Text, p 228, Ex 36)* **and** *(C U Pap 1893)*

** 524 Two pts being given in a given st line, to determine a third, such that the rect contained by its distances from each extremity and the given pt adjacent to that extremity, may be equal*

Let AB be the given st line, C and D the given pts in it On AC and DB as diameter, let ⌒s be described, and let EF touch them in E and F. Bisect EF in G, and let fall the ⊥ GH Then H is the pt reqd From G draw st. lines GNK, GML cutting the ⌒s Find O the centre of ○ ACE (III 1), draw OP ⊥ GK Now $NG\cdot GK+PN^2=PG^2$ (II 6), to each of these equals, add PO^2, ∴ $NG\cdot GK+(PN^2+PO^2)=NG\cdot GK+ON^2$ or $OC^2=PO^2+PG^2=OG^2=OH^2+HG^2$ (I 47) But $OH^2=CH\cdot HA+OC^2$ (II 6), ∴ $NG\cdot GK+OC^2=(CH\cdot HA+OC^2)+HG^2$, $NG\cdot GK=CH\cdot HA+HG^2$ So $LG\cdot GM=DH\cdot HB+HG^2$ Since $GE=GF$, ∴ $GE^2=GF^2$, ∴ $NG\cdot GK=LG\cdot GM$ (III. 36), ∴ $CH\cdot HA=DH\cdot HB$

**525 Through a given pt between two indefinite st lines not ‖ to one another, to draw a st line which shall be terminated by them, so that the rect contained by its segments, shall be < than the rectangle contained by the segments of any other st line drawn through the same point*

Let AB, AC be the given st lines meeting in A. In AC take any pt D, and make AE=AD Join DE, and through I the given pt, draw FIG ||| DE. Then FIG is the st line reqd Draw the ⊥s FO, GO meeting in O. Then since ED ||| FG, and ∠AED= ∠ ADE, ∠ AFG=∠AGF. But ∠AFO= ∠AGO, each being a rt ∠, ∠ OGF= ∠OFG, and OF=OG; ∴ a ◯ described from the centre O, and radius OG, will pass through F, and touch AB, AC in G and F, since ∠s at G and F are rt. ∠s, (III 16) Let any other st line HKLM be drawn through I, and terminated by AB, AC Since all other pts in AB, except G, are without the ◯, H is *without* the ◯, ∴ HM cuts the ◯ in K So, also in L Now KI IL=GI IF (III 35), ∴ the rect GI IF < rect HI IM Similarly, it may be shewn that, the rect GI IF < the rect contained by the segments of any other st line drawn through I, and terminated by AB, AC

* 526 *Having given the radii of two ◯s, which cut each other, and the distance of their centres, to draw a st line of given length, through their pt of intersection, so as to terminate in their ◯ces*

Let the two ◯s AFD, BGD, cut each other in D, on OC the st line joining their centres O and C, describe a semi-◯ CEO and in it from C, place CE=½ the given st line and through D, draw FDG ||| to it, FG will be the st line reqd Through E, draw OEH which will be ⊥ FG (III 31), and CI ||| OH, and ∴ ⊥ DG Then FD and DG are bisected in H and I, (III 3), and ∴ FG=2 HI, but HECI being a ▭m, HI=EC, ∴ FG=2 EC, and hence=the given st line

527 *To describe two ◯s, each having a given radius, which shall touch each other and the same given st line on the same side of it*

Let AB be the given st line From any pt A in it, draw AC at rt ∠s to it, and make AC,AD=the given radii Produce CA to F, making AE=AD Draw DO ||| to AB, and with centre C, and radius CE, describe a ◯ cutting DO in O Then C and O will be the centres of the ◯s required Join CO, and draw OB ⊥ to AB, then ∠ DAB=a rt ∠, also ∠ ABO=a rt ∠, ∴ AD ||| BO, and DO is ||| to AB, ∴ AO is a ▭m, and OB=AD With the centres C and O, and radii CA, OB, describe ◯s, they shall touch AB (III 16), since ∠s at A and B are rt ∠s, they shall also touch each other For CO=CE=CA+AE=CA+AD or=the *sum* of the radii (III 12, Cor)

528 Describe a ◯ to pass through a given pt, and touch a given st. line and a given ◯ *(See Text p 238, Ex 25)*

* 529. *To describe a ○, which shall touch a st. line and two ○s, given in magnitude and position*

Let A and B be the centres of two ○s, and CD the st line, given in position From B, let fall the ⊥ BE, and produce it, making EF=the radius of the ○ whose centre is A Through F, draw FG ∥ to CD. With centre B, and radius=the *difference of the radii of the two ○s*, describe a ○, through A let a ○ be described, touching the st line GF and the *last described* ○ (Text Ex. 25 p 238), and let G and H be the pts of contact The centre of this ○, will also be the centre of the ○ reqd Let O be the centre, join OA, OG, OH, and with the centre O, and radius OI, describe the ○ IKL. Since LG=KH=AI, ∴ OL=OK=OI, ∴ the ○ IKL touches CD in L, and the ○ whose centre is A, in I: and since OB=OH−HB *i e* the difference between OA and (IA−BK) or =OK+KB, ∴ it touches the ○ whose centre is B, in K.

* 530. (*a*) Find the *locus* of pts from which the tangents drawn to two given ○s, are equal.

(*b*) To draw the *radical axis* of two given ○s.

(*c*) The radical axis of three ○s *taken in pairs*, are *concurrent* (*See Text, Appendix, or pp 371-373*)

* 531 *If with any pt in the ○ce of a ○ as centre, and distance from its centre as radius, a* circular arc *be described, and any two chords be drawn, one from the centre of the circular arc, and the other through the point, where this cuts the arc, and ∥l to the st line joining the centres, the segments of each chord intercepted between the ○ces which are* concave *to each other, will be = respectively to those of the other, between the other ○ces*

With any pt C in the ○ce of the ○ ABC as a centre, and radius CE=the distance from the centre E, let a ○ DFE be described Join CE and draw any chord CFA, and through F, draw HFG ∥l CE (I 31), then shall CF be=FH, and GF=FA Produce CE to B, and join HE And ∵ HG is ∥l BC (cons), ∴ ∠FHE=∠HEB (I 27) Since the ○s are equal, arc HB=arc FE (III 27); ∴ ∠HEB=∠FCE, (III 21), ∴ ∠FCE=∠FHE, and HC is a ▭, hence HF=EC=CF. Since CF.FA=HF.FG (III 35), and HF=CF, ∴ FA=FG

* 532 *The base of a rt ∠d △, not being > than the ⊥, if on any st line drawn from the vertex to the base, a semi-○ be described, and a chord=to the ⊥ placed in it, and bisected, the point of bisection will always* fall within *the* △

Let ABC be a rt. ∠d △, of which the side AC is *not* > BC. From B, let any st line BD be drawn to the base; on which describe

a semi-⊙ BCD, and in *it* place EF = BC, which is bisected at G, the pt G is *within* the △ ABC Find O the centre of the semi-⊙, draw OH ⊥ to BC, join OG Since BC = EF, OH = OG, and the ∠s at G and H being rt ∠s, a ⊙ described with the centre O and radius OG, will *touch* BC in H, G is *within* the ∠DBC Also since AC is *not* > BC, DC < BC or EF, ∴ EF is *nearer* to the centre O, than DC is, or G *fall above* DC and *within* the ∠ DCB

*533 *Given the st. line bisecting the vertical ∠, the ⊥ drawn to it from one of the ∠s at the base, and the other ∠ at the base, to construct the △*

Let AB = the given bisecting st line, and on it, describe a segment of a ⊙ containing an ∠ = the given ∠ (III 33) Draw BC ⊥ to AB, and make BD = the given ⊥ Bisect AB in E, join ED, and produce it to F, join FA, FB, and through D, draw GDH ∥ AB In FB produced, take BI = BH Join AI, AFI is the △ reqd Join IG, cutting AB in K Since GH is ∥ AB, and FE bisects AB, it also bisects GH, *i e* GD = DH, but HB = BI, BD is ∥ GI, and IK = ½ IG and = BD the given ⊥ Also, since AB bisects GI at rt ∠s, it bisects the ∠ IAG and it is = the given bisecting st line And ∠AFI = the other given ∠

* 534 Given the base and vertical ∠ of a △, find the *locus* of the *intersection of the medians* (*See Text p 228, Ex 37*)

* 535 *Given one of the ∠s at the base, the side opposite to it, and the rect contained by the base and that segment of it made by the ⊥, which is adjacent to the given ∠, to construct the △*

Let AB = the given side Upon it, describe a segment of a ⊙ containing an ∠ = the given ∠ (III 33) Bisect AB in C, and from C draw to the ⊙ce, a st line CD, such that $CD^2 \sim CB^2$ may be = the *given rectangle* Join AD, DB Then ADB is the △ reqd On AB describe a ⊙ ABE Join BE Then the rect AD DE = rect GD DF = $CD^2 - CG^2$ = the given rectangle And BE is ⊥ to AD, AD = the given side, and ∠ADB = the given ∠

* 536 *Given the segments of the base made by the ⊥, and one of the ∠s at the base* triple *the other, to construct the △*

AB, BC = the given segments, let them be placed in the same st line Make BD = BC, bisect AD in E and BE in F On AD describe a semi-⊙, and from F draw FG at rt ∠s to AD Join AD, GD And let AG meet the ⊥ BH in H Join HC, then AHC is the △ reqd Draw FI ⊥ to AD, join DI, DH Then AE being = ED and the ∠s at E rt ∠s, AI = ID, and ∠ IAD =

$\angle$ IDA; hence $\angle$ DIH $=2\angle$ DAH But, since EF $=$ FB, and GF is ||| to IE and BH, $\therefore$ IG $=$ GH; and the $\angle$s DGI, DGH being rt $\angle$s, DI $=$ DH and $\angle$ DHI $=\angle$ DIH, and $\cdot$ $=2\angle$ DAH Also since DB $=$ BC and $\angle$s at B are rt $\angle$s, $\therefore$ $\angle$ HCB $=\angle$ HDB $=$ $\angle$ DHA $+\angle$ DAH $=3\angle$ DAH Also AB, BC are $=$ the given segments made by the $\perp$, (cons)

* 537 *Given the difference between the segments of the base made by the $\perp$, the sum of the squares of the sides, and the* area. *to construct the* Δ

Take a st line AB, such that its square may be $=$ the difference between $\frac{1}{2}$ *the given sum of the square* and *the square of* $\frac{1}{2}$ *the given difference of the segments of the base* On AB describe a rectangular ▭ BACD $=$ the given *area*, and on AB describe a semi-○, cutting CD in E Join AE, and produce it *both ways*, make AF, AG, each $=\frac{1}{2}$ the given difference of the segments, and make EH $=$ EG Join BF, BH Then BFH is the Δ reqd Join BG, BE Since GE $=$ EH and $\angle$s at E are rt $\angle$s, $\therefore$ GB $=$ BH; and $FB^2+BH^2=FB^2+BG^2=2(FA^2+AB^2)$, (Ex 245) $=$ the given sum, (cons) Also FE $-$ EH $=$ FE $-$ EG $=$ the given difference And the *area* of the Δ FBH $=2\Delta$ ABE (I 41) $=$ ABCD $=$ the *given area*

538 *Given one* $\angle$, *and a st line drawn from one of the other angles bisecting the sides opposite to it, to construct the* Δ, *when the* area *is given*

Let AB be the given bisecting st line, and on it, describe a segment of a ○, containing an $\angle$ $=$ the given $\angle$ (III 33), on AB describe a rectangular ▭ ABCD $=$ the *given area*, and let DC meet the ○ in E. join EA, and produce it, making AF $=$ AE, join FB, BE, then FEB is the Δ reqd For BA bisects the side EF, $\angle$ BEF $=$ the given $\angle$, and Δ BEF $=2\Delta$ BAF (I 38), and $\therefore$ $=$ ▭m ABCD $=$ the given area

539. In an acute $\angle$d Δ. the $\perp$s drawn from the vertices to the opposite sides, bisect the $\angle$s of the ***pedal*** **Δ through which they pass** (*See Text p 225, Ex 20*)

* 540 *Given the* $\perp$, *the st line bisecting the vertical* $\angle$, *and the st line bisecting the base, to construct the* Δ

From any pt. C in the *indefinite* st line AB, draw a $\perp$ CD $=$ the given $\perp$, and with D as centre and radii $=$ the *two* given st lines, describe ○s, cutting AB in E and F Through E, draw GEH $\perp$ to AB, join DE, DF, and produce DF to meet HE in G Bisect DG in I, and draw OI at rt $\angle$s to DG, meeting GH in O With the centre O, and radius OG, describe a ○ cutting AB in K and L Join DK, DL, DKL is the Δ reqd Join OD, OK, OL

Since OI bisects DG at rt ∠s, OD=OG, and the ⊙ passes through D And since OE is ⊥ to KL, KE=EL or KL is bisected by DE, which is=the given bisecting st line, and the arc KG=arc GL, and ∠KDF=∠FDL or ∠KDL is bisected by DF, which is = the given st. line, and DC has been made=to the given ⊥.

541 *Two ⊙s intersect in A and B Two pts C and D are taken on one of the ⊙s, CA, CB meet the other ⊙ in E, F, and DA, DB meet it in G, H, shew that FG is ∥ to EH*

Since ∠CAG=∠CBD (III 21), ∠GAE=∠FBH, ∠GFE =supplement of ∠GAE (III 22)=supplement of ∠FBH= ∠HGF, and ∠GEF=∠GHF (III 21), ∠EGF=∠GFH, ∴ ∠GFH+∠FHE=2 rt ∠s (III 22), GF is ∥ to EH

542 *From each angular pt of a Δ, a ⊥ is let fall upon the opposite side, prove that the rectangles contained by the segments, into which each ⊥ is divided by the pt of intersection of the three, are=to each other*

Let ABC be a Δ, AD, BE, CF the ⊥s from the angular pts on the opposite sides, intersecting in O Then a ⊙ described on BC as diameter, will pass through F and E (III 31), ∴ rect CO OF=rect BO OE (III 36) Again a ⊙ described on AB as diameter, will pass through E and D, ∴ rect BO OE=rect AO OD (III 36)

543 *Produce a given st line AB, so that the rect contained by the whole st line thus produced, and part of it produced, shall be =to a given square* (Punjab Ex Pap 1886)

Bisect AB in C, on AB as diameter, describe a ⊙ ADB, from any pt. D in the ⊙ce of ⊙ ADB, draw the tangent DE=a side of the given square (III 17 and I 2) Join EC From centre C and radius CE, describe a ⊙ cutting AB produced in F, then F shall be the external pt reqd Draw FH tangent to the ⊙ ADB (III 17), and join CH, ∵ ∠s at D and H are rt ∠s (III 18), $ED^2+DC^2=EC^2=CF^2=FH^2+CH^2$, and $DC^2=CH^2$, ∴ DC=CH, ∴ $ED^2=FH^2$ But rect AF FB=FH^2 (III 36), rect AF FB=ED^2=the given square

544 *A ladder is gradually raised against a wall, find the locus of its middle points*

Let O be the middle pt of the *ladder* AB, DC the vertical wall, EC the horizontal plane Then ∵ ∠ACB is a rt ∠, a ⊙ described with centre O and radius OA, will pass through C (III 31), ∴ the locus of O, is the quadrant of a ⊙ described with centre C, and radius OA

545. DR is a diameter of a ⊙. DP, DQ are two chords meeting the tangent at R (i. e. RST) at S and *T respectively. Shew that ∠ TPS = ∠ TQS* (Punjab Ex. Pap 1887)

Join PR and QR, ∠ DPR and ∠ DQR, each = a rt. ∠ (III. 31), for DR is the diameter of the ⊙. ∴ RP and RQ are ⊥s to DS and DF respectively. Now since DR is diameter, and RST is a tangent at R, ∴ ∠ DRS or ∠ DRT is a rt. ∠ (III 16). ∴ △s DRS and DRT are rt. ∠d △s; hence DS.DP = DR² (Ex 237) and DT.DQ = DR² (Ex 237), ∴ DS.DP = DR² = DT.DQ (Ax 1), hence P, S, T, Q are *concyclic* (III 35. *Con.*). ∴ ∠TQS = ∠TPS (III. 21)

* *546. AB is the diameter of a semi-⊙. P is a pt on the ⊙ce, PM is a ⊥ to AB, and on AM and BM as diameters, two semi-⊙s are described and AP, BP meet these latter ⊙ce at Q and R; shew that QR will be a common tangent to them*

Bisect BM at O and join OR and MR, then ∠s APR, MRB are rt. ∠s (III 31), AP and MR are ∥ (I 28). Similarly PR and QM can be proved to be ∥. In the △s QMR, PMR; QM = PR, MR is common, ∠ QMR = ∠ PRM, ∴ ∠ QRM = ∠ PMR (I 4) Also ∠ ORM = ∠ OMR (I 5) hence ∠ ORQ = ∠ PMO = a rt ∠ (const.); and ∴ QR *touches* the ⊙ MRB (III 16) So, it may be proved that QR will touch the ⊙ AQM.

547. Two ⊙s touch internally at A. A st. line touches one ⊙ (i e the inner*) at P, and cuts the other (i e the* outer ⊙*) at Q and R. Prove that PQ and PR subtend equal ∠s at A* (Punjab Ex. Pap. 1881 and 1884, q 6)

Draw AT the common tangent at A (III. 17) Join AQ, AP and AR. To show that ∠ PAQ = ∠ PAR Let AR intersect the *inner* ⊙ at X. Join PX Now from the greater ⊙, ∠ TAQ = ∠ ARQ or ∠ *ARP* (III 32), from the inner ⊙, ∠ TAP = ∠ AXP (III 32); hence ∠ PAQ = ∠ TAP − ∠ TAQ = ∠ AXP − ∠ ARP = ∠ XPR (I. 32) But ∠ XPR = ∠ XAP or ∠ *PAR* (III 32), ∴ ∠ PAQ = ∠ PAR, *i. e* PQ and PR subtend equal ∠s at A

548. To find at what pt in a given st line, the ∠ subtended by the st. line joining two given pts which are on the same side of the given st line, is a *maximum*. (See Text p 242)

549 If any st. line be drawn ⊥ to the diameter of a given ⊙, and produced to cut any chord, the rect. contained by the segments of the diameter will be < or > than the rect. contained by the segments of the chord, by the square of the st. line intercepted between them, according as it is drawn without *or* within *the* ⊙.

Let AB meet the diameter CD of the ◯ CGD at rt ∠s in the pt E, and any other chord DH in F The rect CE ED is $<$ or $>$ the rect GF FH, by EF^2, according as AB is *without or within* the ◯ Find O the centre of the ◯ (III 1), and through it, draw FOK cutting the ◯ in I and K Then ∵ KI is bisected in O, and produced to F, the rect $KF\,FI + OI^2 = OF^2$ (II 6) $= OE^2 + EF^2$ But when E is *without* the ◯, the rect $CE\,ED + OD^2 = OE^2$ (II 6), ∴ the rect $KF\,FI + OI^2 = CE.ED + OD^2 + EF^2$ And since $OI = OD$, and the rect $KF\,FI = GF\,FH$ (III 36, Cor), ∴ the rect $GF\,FH = CE\,ED + EF^2$ So, it can be proved, if AB be *within* the ◯.

BOOK IV

550 In the Δ XYZ, A and B are the centres of the *inscribed* and *circumscribed* $\bigcirc$s, shew that AB subtends at X an $\angle$ = to *half* the difference of the $\angle$s at the base of the Δ.

Join XB, ZB, XA. Since XB = BZ, $\therefore$ $\angle$ BXZ = $\angle$ XZB (I. 5), but $\because$ $\angle$ XBZ = 2 $\angle$ XYZ (III. 20), and $\therefore$ $\angle$XBZ+($\angle$ BXZ+$\angle$XZB)=2 rt. $\angle$s=2 $\angle$ XYZ+2 $\angle$ BXZ, $\therefore$ $\angle$ BXZ + $\angle$Y= a rt. $\angle$, $\therefore$ $\angle$ BXZ a rt. $\angle$ — $\angle$ Y. Hence $\angle$ BXA = ($\angle$ *ZXA* − $\angle$ BXZ)=$\frac{1}{2}$ $\angle$ *X* − (a rt. $\angle$ − $\angle$Y), (IV. 4) =$\frac{1}{2}$ $\angle$X −($\frac{1}{2}$ X+$\frac{1}{2}$Z+$\frac{1}{2}$Y)+ $\angle$ Y, (I. 32)=$\frac{1}{2}$ $\angle$Y −$\frac{1}{2}$ $\angle$Z

551. Shew that the *area* of a Δ, is = to the rect. contained by its semi-perimeter and the radius of the inscribed $\bigcirc$.

Let XYZ be any Δ, inscribe a $\bigcirc$ in it (IV. 4), let A be the centre of the inscribed $\bigcirc$, touching the sides XY, YZ, ZX at P, Q, R. Join XA, YA, ZA, PA, QA, RA. The *area* of the Δ XYZ = Δ XAY + Δ YAZ+ Δ XAZ =$\frac{1}{2}$ XY.PA +$\frac{1}{2}$ YZ.QA + $\frac{1}{2}$ZX.RA, (I. 41)=$\frac{1}{2}$ (XY+YZ+ZX) PA, (II. 1), for AP=AQ=AR.

552. PQRS is a square inscribed in a $\bigcirc$, and X is any point on the $\bigcirc$ce, shew that $XP^2+XQ^2+XR^2+XS^2=2$ (diameter)^2.

Join PR, QS intersecting at O. Then O is the centre of the $\bigcirc$ (IV. 6). Join OX. Since $XP^2+XR^2=2\ PO^2+2\ OX^2$ and $XQ^2+XS^2=2\ OQ^2+2\ OX^2$ (Ex. 245), $\therefore$ $XP^2+XQ^2+XR^2+XS^2=4\ PO^2=4\ XO^2$, for PO=QO, $\therefore$ $XP^2+XQ^2+XR^2+XS^2=$ (diameter)^2+(diameter)^2=2 (diameter)^2

553. In an isosceles Δ, in which each of the $\angle$s at the base=2 vertical $\angle$ (as in IV. 10), shew that the vertical $\angle$=$\frac{1}{5}$th of 2 rt. $\angle$s. (Cal. Ex. Pap. 1878)

Since each of the $\angle$s at the base is =2 vertical $\angle$, $\therefore$ the sum of the $\angle$s=5 times the vertical $\angle$. Hence 5 times the vertical $\angle$=2 rt. $\angle$s (I. 32); and $\therefore$ the vertical $\angle$ =$\frac{1}{5}$th of 2 rt. $\angle$s.

554. To divide a rt. $\angle$ into *five* equal parts. *(Allahabad Ex. Pap. 1890, q. 10)*

Let ABC be a rt. $\angle$. Draw BX (I. 23) making $\angle$ABC=the vertical $\angle$ of an isosceles $\angle$, having each of its base $\angle$s=2 vert. $\angle$ (as in IV. 10). Bisect $\angle$ ABX by BY (I. 9). Draw BZ, BN

making ∠s XBZ, ZBN each = ∠XBY (I 23) Then ∠ABC is divided into 5 equal parts by BY, BX, BZ, BN Since ∠ABX = $\frac{1}{5}$th of 2 rt ∠s, ∴ each of the ∠s ABY, YBX = $\frac{1}{5}$th of a rt ∠ (Cons) Also, since each of the ∠s XBZ, ZBN = ∠XBY (con), ∴ the whole ∠ABN = $\frac{4}{5}$th of a rt ∠, ∴ the remaining ∠NBC = $\frac{1}{5}$th of a rt ∠

555 *The sum of the diameters of the inscribed and circumscribed ○s of a rt ∠d Δ QPR = the sum of the sides QP, PR containing the rt ∠*

Let PQR be the Δ, rt ∠d at P, inscribe a ○ XYZ to touch the sides QP, PR, RQ at X, Y, Z (IV 4), let A be its centre Join AX =, AY, then the ∠s at X and Y are rt ∠s (III 18) and AX, AY, ∴ the fig PXAY is a square Hence the diameter of the inscribed ○ = XA + AY Again ∵ ∠QPR = a rt ∠, ∴ the diameter of the circumscribed ○ = QR (III. 31) = RZ + ZQ = RY + QX (III 17, Cor) Hence, the sum of the diameters = (XA + AY) + (QX + RY) = QP + PR

556 *In any Δ, the difference of two sides, is = to the difference of the segments, into which the third side is divided, at the point of contact of the inscribed ○*

Let PQR be any Δ, in Δ PQR, inscribe a ○ XYZ (IV 4), let X be the pt where the third side QR touches the inscribed ○ Then shall PQ ∼ PR be = QX ∼ RX Let PQ and PR touch the inscribed ○ at Y and Z Since PY = PZ (III 17, Cor), PQ ∼ PR = (PY + QY) ∼ (PZ + RZ) = QY ∼ RZ, but QY = QX and RZ = RX (III 17, Cor) ∴ PQ ∼ PR = QX ∼ RX

557 An equilateral Δ is inscribed in a ○, and tangents are drawn at its vertices, prove that (i) the resulting figure is an equilateral Δ, (ii) its area is = ***four*** **times that of the given Δ**

Let XYZ be the inscribed equilateral Δ, and PQ, PR, RQ the tangent through the pts X, Y, Z forming the Δ PQR by their intersections (1) Each of the ∠s PXY, PYX = ∠XZY (I 32) = ⅓ of two rt ∠s (hyp), ∴ ∠P = ⅓ of 2 rt. ∠s (I 32) So for the ∠s Q and R, ∴ the Δ PQR is equiangular and hence equilateral (I 6, Cor) (2) ∠PXY = ∠XZY (III 32) and ∠PYX = ∠XZY (III 32) = ∠XYZ (hyp), and XY = YZ (hyp), ∴ ΔPXY = Δ XYZ (I 26) So, Δ QXZ = ΔYZR = ΔXYZ, ∴ the area of ΔPQR = 4 times the area of Δ XYZ

558 *If the ○ inscribed in a Δ ABC, touch AB, AC in D, E, and st line AO be drawn from A to the centre O of the ○, meeting the ○ce in G, shew that G is the centre of the ○ inscribed in Δ ADE.* **(Cam Ex Pap 1855).**

Join OD, OE, DE, DG, EG. Draw GK, GH $\perp$s AD, AE. In Δs ADO, AEO, AD=AE (III. 17, Cor.), OD=OE, AO is common, ∴ ΔDAO=ΔEAO (I. 8). Let AO cut DE at F. In Δs DAF, EAF, $\angle$DAF=$\angle$EAF (proved), $\angle$ADF=$\angle$AEF (I. 5) and AD=AE, ∴ AF bisects DE at rt. $\angle$s (I. 26). In Δs DFG, EFG, DG=EG, and $\angle$GDF=$\angle$GEF, but $\angle$ADE=$\angle$AED, and ∵ their parts, $\angle$GDF and $\angle$GEF are equal, ∴ the rem. $\angle$ADG or $\angle$KDG=$\angle$AEG or $\angle$HEG. From Δs KDG and HEG, it can be shewn, that GK=GH. Again, $\angle$HEG=$\angle$GDE (III. 32)=$\angle$GEF, ∴ from Δs GHE, GFE, GF=GH, ∴ a ○ described with G as centre, and GF or GK or GH as radius, will touch the sides of Δ ADE.

559. Describe a ○, to touch two ∥l st. lines and a third st. line, which meets them. Shew that, two such ○s can be drawn, and that they are equal. (Bom. Ex. Pap. 1885.)

Let PQ, RS be the two ∥ st. lines cut by XY. Bisect $\angle$s PXY, RYX by st. lines which meet at A. Then A is the centre of the reqd. ○. Draw AB, AC, AD $\perp$s to PQ, RS, XY. Since $\angle$ BXA=$\angle$ DXA (cons.), and $\angle$ XBA=$\angle$ XDA (a rt. $\angle$), and AX is common to the two Δs BXA, DXA, ∴ AB=AD, so AD=AC (I. 26). Hence, AB=AC=AD. And since the $\angle$s at B, C, D are rt. $\angle$s, a ○ described with centre A and radius AB touches the given st. line at B, C, D (III. 16). The centre of the *2nd* ○ is obtained by bisecting the $\angle$s QXY, SYX which meet in Z. Draw ZO, ZE $\perp$s to PQ, RS. Since AB, AC are both $\perp$s to ∥ st. lines, ∴ they are in the same st. line. So, ZO, ZE are in the same st. line. Hence BC, OE are the diameters of the two ○s. But BC=OE (I. 34), ∴ the ○s are equal.

560. In IV. 10, if A be the vertex, BD the base of the constructed isosceles Δ, D, E the pts. of intersection of the two ○s, and AE be drawn meeting BD produced in F, prove that Δ FAB is another isosceles Δ, *of the same kind*. (Cam. Ex. Pap. 1860.)

$\angle$AED = supplement of $\angle$ACD (III. 22)=$\angle$BCD = $\angle$ABD, ∴ $\angle$ ADE = $\angle$ AED = $\angle$ABD = $\angle$ ADB, ∴ rem. $\angle$ DAE = rem. $\angle$ BAD, ∴ $\angle$BAF = 2 $\angle$ BAD=$\angle$ABD, and $\angle$ ABF= 2 $\angle$ BAD=$\angle$ADB, ∴ $\angle$ AFB=$\angle$BAD.

561. If the inscribed and circumscribed ○s of a Δ, are *concentric*, shew that (1) the Δ is equilateral; and (2) the diameter of the circumscribed ○, is *double* that of the inscribed ○. (Bom. Ex. Pap. 1880).

(1) Let XYZ be any Δ, and O the centre both of the inscribed

and circumscribed ◯s Then shall △ RYZ be *equilateral* Join OX, OY, OZ Then OX=OY=OZ (IV 5), and OX, OZ are the bisectors of ∠s YXZ, XZY respectively (IV 4) Since OX=OY, ∠ OXY=∠ OYX (I 5) But ∠ YXZ=2 ∠ OXY, and ∠XYZ=2∠OYZ (hyp), ∴ ∠YXZ=∠XYZ So, ∠XZY=∠YXZ=∠XYZ, ∴ the △ XYZ is equilateral

(2) Produce ZO to meet XY at Q Then ZQ bisects XY at rt ∠s Hence OQ is a radius of the inscribed ⊙ So YO produced, bisects XZ at rt ∠s, ∴ O is the pt of intersection of the *medians*, and of the ⊥s Hence OZ=2 OQ, ∴ 2 OZ (diameter of the circumscribed ◯)=2 (2 OQ) or 2 (diameter of the inscribed ◯)

562 Describe a ◯ touching one side of a △, and the produced parts of the other two (See Text p 255) (Cam Ex Pap 1849, 1861, Bom Ex. Pap 1863)

563 In a given ◯, inscribe a △, whose ∠s are as the numbers 2, 5, 8 (Cam Ex Pap 1851)

Inscribe a regular *quindecagon* in a ◯ (IV 16), each of the ∠s of this fig $=\frac{26}{15}$ of one rt ∠ $=\frac{13}{15}$ of 2 rt ∠s (I 32, Cor I) If now, we produce a side of the quindecagon, we see (that the ∠ between the produced part and the adjacent side + one interior ∠ of the fig or $\frac{13}{15}$ of 2 rt ∠s)=2 rt ∠s (I 13), ∴ the ∠ between the produced part and the adjacent side=$\frac{2}{15}$ of 2 rt ∠s (*a*) Take another ∠ = 4 times the ∠ mentioned in (*a*) *i e* and ∠ $=\frac{8}{15}$ of 2 rt ∠s (*b*) Also take an ∠ of an equilateral △ *i e*, an ∠ $=\frac{5}{15}$ of a rt ∠s (*c*) Construct a △ with its ∠s= the ∠s mentioned in (*a*), (*b*), (*c*), and inscribe in the given ◯ a △, having its ∠s=those mentioned in (*a*), (*b*), (*c*)

564 If the circle inscribed in a △ ABC, touches the sides at D, E, F, shew that the △ DEF is acute-∠d, and express its ∠s in terms of the ∠s at A, B, C (*Cal Ex Pap 1868*)

Inscribe a ◯ DEF in △ ABC (IV 4), touching AB, BC, CA in D, E, F, ∴ AD=AF (III 17), ∴ ∠ADF=∠AFD (I 5) Hence, each of the ∠s ADF, AFD is *acute*. But ∠ADF=∠DEF (III 32), ∴ ∠DEF is *acute* So, ∠s EDF, DFE may be shewn to be *acute*, ∴ △ DEF is acute-∠d

565 In the figure of IV 10, if the two ◯s intersect at E, shew that BD=DE

Join AE. Since AD=AE, ∠ADE=∠AED (I 5), also ∠ADB=∠AED (III. 32) Hence, in the two △s ABD, AED, two ∠s of the one=two ∠s of the other, and AD is common; ∴ BD=DE (I. 26)

566. Construct an isosceles Δ, whose exterior vertical $\angle = 67\frac{1}{2}$ degrees. *(Cal Ex Pap., 1862).*

Take a st line PS Draw PY at rt $\angle$s to it Bisect $\angle$ SPY by PX and $\angle$ SPX by PR. In YP produced, take PQ=PR, and join QR Then PQR is the reqd Δ, $\because \angle$ YPX=45° (cons), and $\angle$XPR=22½° (cons.), $\therefore \angle$ YPR=67½° Also PR=PQ (cons), hence Δ PQR is isosceles

567 In the figure of IV 10, if the two ○s intersect at E shew that (1) BD, DE are sides of a regular *decagon* **inscribed in the ⊙ PBD** *(Cal Ex Pap, 1878)*

(2) **Also shew that AC, CD, DE are the sides of a regular** *pentagon*, **inscribed in the ○ ACD** *(Cam Ex. Paps, 1850, 1875)*

(1) Since $\angle$ BAD=⅕ of 2 rt $\angle$s=$\frac{1}{10}$ of 4 rt $\angle$s, BD subtends at the centre of the ○ PBD, an $\angle$ =$\frac{1}{10}$ of 4 rt $\angle$s; and hence, the side of a regular *decagon* inscribed in the ⊙ But BD=DE (Ex 565), $\therefore$ BD, DE are the sides of a regular *decagon* inscribed in the ○ PBD

Since $\angle$ CAD=⅕ of 2 rt. $\angle$s =$\frac{1}{10}$ $\angle$ of 4 rt $\angle$s.

(2) $\angle$ CAD=½ the $\angle$ at the centre of the ○ ACD (III 20), $\therefore$ CD subtends at the centre of the ○ ACD, an $\angle$ =⅕ of 4 rt $\angle$s, and hence a side of a regular *pentagon* inscribed in the ○ ACD But CD=AC (proved)=DE, $\therefore$ AC, CD, DE are the sides of a regular *pentagon* inscribed in the ○ ACD.

568 Two st lines OA, OB being given intersecting in O, and a point C being given in OA, describe a ○ touching OA in C and also touching OB (Cal. Ex. Pap, 1870)

Bisect $\angle$ AOB by OF From C draw CE at rt. $\angle$s to OA, cutting OF in E Then E is the centre of the reqd ○ From E, draw ED ⊥ to OB Since $\angle$COE= $\angle$ DOE (cons); $\angle$OCE= $\angle$ ODE (a rt $\angle$), and OE is common to two Δs COE, DOE, $\therefore$ CE=DE (I 26) Hence, the ○ described with centre E and radius EC, will pass thro' D Again $\because$ OD and OC are ⊥s to DE, CE, $\therefore$ OD, OC touch the ⊙ at D and C (III. 16)

569 Describe a ○ and a *square* **about a given rect**

(i) Let PQRS be the given rect, join PR, QS, intersecting at O Then shall O be the centre of the reqd ○. Since PR =QS, and they bisect each other at O, $\therefore$ OP=OR=OQ=OS Hence, the ⊙ with centre O and radius OP, will pass thro' Q, R and S

(ii) On PQ, SR, as hypotenuse, describe two rt. $\angle$d isosceles Δs PQX, SZR *outside* the rectangle Produce XQ, ZR to meet

at Y, and XP, ZS to meet at A. Then shall XYZA be the reqd. square, ∴ XP=XQ (cons), ∴ ∠ XPQ= ∠ XQP (I. 5), but ∠QXP=a rt ∠, ∴ ∠ XPQ=½ a rt ∠. And ∠ QPS=a rt ∠ (hyp), ∠APS=½ a rt ∠ (I. 13). So, ∠ASP=½ a rt ∠, and ∴ ∠ PAS=a rt ∠. Hence Δ PAS is a rt ∠d isosceles Δ, on PS. So, it may be shewn that QYR is a rt. ∠d isosceles Δ on QR. Again, Δ XPQ= Δ RSZ (I. 26), Δ PAS= Δ QYR, the fig. YA is *equilateral*. Hence it is a square.

570. Two equilateral Δs XYZ, PNO are described about the same ◯. Shew that their intersections will form a *hexagon* equilateral, but not generally equiangular. *(Cam. Ex. Pap., 1852.)*

Let Q, M, B, denote the pts. of intersections of the sides, and R, A two of the pts. of contact. Now *RN*=½ PN=½ XY=*XA*, or MR+MN=XM+MA, but MR=MA (III. 17, Cor.), MN= XM. In Δs XQM and MBN, ∠XMQ= ∠ NMB (I. 15), ∠X= ⅓ of 2 rt ∠s= ∠ N, and XM=MN, ∴ MQ=MB (I. 26). So, the other sides of the *hexagon* may be shewn to be equal. When PN is ||l to YZ, ∠ XQM= ∠XMQ, ∴ ∠MQY= ∠QMB and the *hexagon is* equiangular, it is not equiangular in any other case.

571. The square on the side of an equilateral Δ described about the ◯, is four times the square on the side of an equilateral Δ inscribed in the same ◯. (Cal. Ex. Pap., 1880.)

Let PQR be an equilateral Δ, inscribed in a ◯. At P, Q, and R, draw tangents to the ◯, forming by their intersections the Δ XYZ. Then XYZ is an equilateral Δ described about the ◯. Since ∠PQX= ∠PRQ (III. 32)=⅓ of 2 rt ∠s= ∠PXQ, ∴ PX=PQ (I. 6). So PR=PY, but PQ=PR (hyp.), PX=PY and XP=2 PQ. Hence $XY^2=4\ PQ^2$.

572. In any Δ XYZ, if P be the centre of the inscribed ◯, and if XP is produced to meet the circum-◯ at A, shew that A is the centre of the ◯ circumscribed about the Δ YPZ.

Join YA, ZA, YP, ZP. Since ∠ZXA= ∠YXA (IV. 4), ∴ arc ZA =arc YA (III. 26), ∴ chord ZB=chord YA (III. 29). Since the ext. ∠ZPA=(∠ZXP+ ∠PZX) (I. 32)=½ (∠X+ ∠Z), (IV. 4); and ∠ PZA= ∠ PZY+ ∠ YZA= ∠ PZY+ ∠ YXA, (III. 21)= ½ ∠Z+½∠X, ∠ZPA= ∠PZA. Hence ZA=AP (I. 6), PA=ZA =YA, and A is the centre of the ◯ about the Δ YZP.

573. In an equilateral Δ, the radii of the circumscribed and escribed ◯s are respectively *double* and *treble* of the radius of the inscribed ◯.

Let PQR be an equilateral Δ, C be the centre of the inscribed $\odot$, and D the centre of the *escribed* $\bigcirc$ touching PQ. Then C is also the centre of the circum-$\odot$, and the intersection of the *medians* Let PCD cut QR at O Then CO, CP and DO are the radii of the *inscribed*, *circumscribed* and *escribed* $\bigcirc$s And CP $=2$ CO Join DQ Since QD is the bisector of the external $\angle$ at Q, $\therefore$ $\angle$ DQR $=$ ⅓ of 2 rt $\angle$s$=$ $\angle$ PQR, and $\angle$ DOQ$=$ $\angle$ POQ, also QO is common to the two Δs DOQ, POQ, $\therefore$ DO$=$PO (I 26) Hence EO$=$PC$+$CO$=$2CO$+$CO$=$3CO.

574 *Shew how to derive a* **regular hexagon** *from an equilateral Δ inscribed in a $\bigcirc$, and from the construction shew, that the side of the hexagon, equals the radius of the* $\odot$, *and that the* **hexagon** *is double of the* Δ (Cam Ex Pap, 1856, *Cf* Cal. Ex. Pap, 1872)

Let PQR be an equilateral Δ inscribed in a $\odot$, whose centre is A From A, draw AB $\perp$ to QR, and produce it to meet the $\bigcirc$ce at S Join QS, SR Then QS, SR are the sides of a regular *hexagon* inscribed in the same $\bigcirc$ Join PA, QA, RA, $\because$ QA $=$RA, AB is common, and QB$=$RB (III 3), $\angle$ QAB$=$$\angle$ RAB or AB bisects the $\angle$ QAR, $\because$ $\angle$ QAB $=$ $\angle$ QPR (III 20). Thus $\angle$ QAS is an $\angle$ of an equilateral Δ But since AQ$=$AS, $\therefore$ $\angle$ AQS$=$$\angle$ ASQ Hence, each $\angle$ of the Δ QAS is a Δ of an equilateral Δ, and QS $=$ SA So, it may be shewn that, each side of the *hexagon* is$=$the *radius* of the given $\odot$; and each $\angle$ of the *hexagon* is$=$*twice* the $\angle$ of an equilateral Δ Hence, the hexagon is regular Again $\because$ Δ QAR$=$ ΔQSR, as also Δ PAR$=$$\Delta$ PRY, and Δ PAQ$=$$\Delta$ PXQ, $\therefore$ the hexagon PXQSRY$=$2 Δ PQR

575. In a *polygon of* (n *sides*), st lines which join any angular pt to the vertices *not* adjacent to it; divide the angle into ($n-2$) equal parts

Let P, Q, R, S, X, Y, Z &c, be the *successive* angular pts of a regular polygon of n sides; join PR, PS, PX, PY &c Then shall PR, PS, PX, PY etc, divide the $\angle$ P into ($n-2$) parts Circumscribe a $\bigcirc$ about the polygon, $\because$ the ($n-2$) sides QR, RS, SX, XY, YZ etc., are equal (hyp), the ($n-2$) arcs QR, RS, SX, XY, YZ etc, are equal (III 28), $\therefore$ the ($n-2$) $\angle$s at the $\bigcirc$ce *i e*, $\angle$ QPR, $\angle$ RPS, $\angle$ SPX, $\angle$ XPY, $\angle$ YPZ ect, are equal (III 27)

576 Shew how to construct on a given st. line (a) a regular *pentagon*, (b) a regular *hexagon*, (c) a regular *octagon*

(1) Let PQ be the given st line Describe an isos. Δ XYZ having each of the base $\angle$s$=$2 the vert $\angle$ (IV. 10) On PQ

as base, describe a Δ PQR *equiangular* to the Δ XYZ (I 23) About Δ PQS, describe a ○ (IV. 5), and bisect the $\angle$s at P and Q by PR and QA meeting the ○ce at R and A Join PA, AS, SR, RQ Then PQRSA shall be the regular *pentagon* on PQ The *proof* is similar to (IV 11)

(2) On PQ describe an equilateral Δ PQR From centre O with radius OP, describe a ○, which will also pass thro' Q Produce PQ, QO to meet the ○ce at S and A Bisect $\angle$ SQO by the diameter RB, and join QR, RS, SA, AB and BP Shen PQRSAB is the regular *hexagon* on PQ reqd *Proof* is the same as (IV 15)

(3) At P, Q make $\angle$s QPO, PQO each $= \frac{3}{4}$ rt $\angle$, meeting in O From centre O with radius OP, describe a ○, which will pass thro' Q Produce PO, QO to meet the opposite ○ce at X and Y Draw the diameter RZ, SA at rt $\angle$s to PX, QY Join QR, RS, SX, XY, YZ, ZA and AP Then PQRSXYZA is the regular *octagon* on PQ Since each of the $\angle$s OPQ, OQP $= \frac{3}{4}$ of a rt $\angle$ (cons), $\therefore \angle$POQ $= \frac{1}{2}$ a rt $\angle$, (I 32) But $\angle$POR is a rt $\angle$, $\therefore \angle$ QOR $= \frac{1}{2}$ a rt $\angle$. So the other *six* $\angle$s at O are each $= \frac{1}{2}$ a rt $\angle$, and $\therefore$ all are equal Hence the *8 arcs* PQ, QR, RS etc, are all equal (III 26), and $\therefore$ *8 chords* PQ, RS, QR, *i e* are all equal (III 29), $\therefore$ the *inscribed octagon* PQRSXYZAB is equailteral, and $\therefore$ equiangular

577 If the st lines which bisect the $\angle$s of a rectilineal figure, are *concurrent*, a circle may be inscribed in the figure

Let PQ, QR, RS etc are the sides of the fig and O the pt of intersection of the bisectors of the $\angle$s Draw OA, OB, OC etc $\perp$s to PQ, QR, RS etc Then since $\angle$ AQO $= \angle$ BQO, $\angle$ QAO $= \angle$ QBO, and QO is common to the two Δs AOQ, BOQ, $\therefore$ OA$=$OB (I 26) So OB$=$OC, and *so on for each pair of sides* Hence $\perp$s OA, OB, OC etc are all equal Thus a ○ may be inscribed in the figure

578 An equilateral Δ PRX and a regular *hexagon*, are inscribed in a given ○ PQRSXY, shew that (i) the area of the $\Delta = \frac{1}{2}$ that of the *hexagon*, (ii) the square on the side of the $\Delta = 3$ times the square on the side of the *hexagon* (*Cf Bom Ex Pap*, *1862*)

(1) Take O the centre, and join OP, OR, OX Since PQ, QR $=$PO, OR (IV 15, Cor), and PR is common, ΔPQR$=\Delta$POR (I. 8) So Δ RSX$=\Delta$ ROX, Δ RYX $=\Delta$ POX. Hence the *hexagon*$=$twice the equilateral Δ PRX

(2) Produce PO to meet RX at Z Then since $\angle$ PRX is equilateral, $\therefore$ PX is $\perp$ to RX; and O is the intersection of the

medians Hence PO=2OZ (Ex 4, Cor p 105, Text); ∴ $PO^2 = 4OZ^2$ Now $PR^2 = PO^2 + RO^2 + 2PO\ OZ$ (II 12) $= PO^2 + RO^2 + PO^2 = 3PO^2 = 3\ PQ^2$ (IV 15, Cor.)

***579 X is any point on the ◯ce of a ◯, described about an equilateral △ PQR, shew that ($XP^2 + XQ^2 + XR^2$) is constant**

Let O be the centre of the ◯ Then O is the *ortho-centre* and *centroid* of △ PQR Let A = radius of the circum-◯ Join PO Produce it to meet QR at B, and the ◯ce at Y. Join XO, XB, and XY Then OY = OP = 2BO (Ex 4, Cor p 105, Text), ∴ OB = BY In △ PXY, $XP^2 + XY^2 = 2OP^2 + 2\ OX^2$ (Ex. 245) $= 4A^2$, and in △ QXR, $XQ^2 + XR^2 = 2QB^2 + 2XB^2$ (Ex 245), ∴ $XP^2 + XQ^2 + XR^2 + XY^2 = 4A^2 + 2QB^2 + 2XB^2$ (1) Also from △ OXY, $OX^2 + XY^2 = 2OB^2 + 2XB^2$ (Ex. 245), and OX = A; ∴ $A^2 + XY^2 = 2OB^2 + 2XB^2$ (2), ∴ by taking (2) from (1), $XP^2 + XQ^2 + XR^2 - A^2 = 4\ A^2 + 2QB^2 - 2OB^2$ To each of these *equals*, add A^2 or $4\ OB^2$, $XP^2 + XQ^2 + XR^2 = 4A^2 + 2QB^2 + 2OB^2 = 4A^2 + 2A^2$ (I 47) $= 6A^2$ = a constant

580 If a rectilineal figure of an *even* number of sides, be inscribed in a ◯, the *first*, *third*, *fifth* &c ∠s are together = to *2nd*, *4th*, *6th*, &c ∠s taken together; any ∠ being assumed as the first (*Cal Ex. Pap*, *1865*)

Let PQRSXY be any rectilineal fig of an *even* number of sides, inscribed in a ◯ Then shall ∠ P + ∠ R + ∠X be = ∠Q + ∠ S + ∠Y Join RY, ∵ ∠ YPQ + ∠ YRQ = 2 rt ∠s (III 22) = ∠PQR + ∠PYR, and ∠ YRS + ∠ YXS = 2 rt ∠s (III 22) = ∠RSX + ∠RYX Hence by addition, ∠P + ∠R + ∠X = 4 rt ∠s = ∠ Q + ∠ S + ∠Y

581 In the figure of IV 10, shew that the centre of the ◯ circumscribed about the △ DBC, is the middle point of the arc CD

Bisect the arc CD at P (III 30), and join DP, CP, AP Produce DP to meet BC at Q and AP to meet BD at R Then arc CP = arc DP, (cons), ∴ ∠ CAP = ∠ DAP (III 27) or AP bisects ∠CAD Hence AR bisects BD at rt ∠s Also ∵ ∠BAD = ∠ CDB and ∠ CAP = ∠ CDP (III 21), DP bisects ∠BDC Hence DQ bisects BC at rt ∠s, P is the centre of the ◯ about △ BDC (IV 5)

582 ABC is an isosceles △, in which each of the ∠s at B and C = 2 ∠A, shew that $AB^2 = AB.BC + BC^2$

Bisect ∠ACB by CX, meeting AB at X, and about △ AXC, describe a ◯ (IV 5) Then, since ∠ BCX = ½ ∠ ACB = ∠ BAC,

∴ BC is a tangent to the ⊙ AXC at C. Hence $BC^2 = AB.BX$ (III 36) Also $AX = XC = BC$ Now, ∵ $AB^2 = AB.BX + AB.AX$ (II 2) $= BC^2 + AB.BC$.

583 A ○ is described so as to touch the sides BC of a △ ABC in D, AB produced in E, and AC produced in F Shew that the △ EDF is obtuse ∠d. (*Cal Ex Pap*, *1874*)

Since BE = BD (III 17, Cor.). ∠ BED = ∠ BDE (I. 5) But ∠ ABC = ∠ BED + ∠ BDE (I 32) = 2 ∠ BDE. So ∠ ACB = 2 ∠ CDF, ∴ ∠ ABC + ∠ ACB = 2 (∠ BDE + ∠ CDF) But ∠ ABC + ∠ ACB is < than 2 rt ∠s (I 17) ∴ ∠ BDE + ∠ CDF is < than a rt ∠ Hence ∠ EDF is > than a rt. ∠ (I 13); ∴ ∠ EDF is an obtuse ∠, ∴ △ EDF is obtuse ∠d.

584 All *cyclic* parallelograms, are rectangular

Let XYZP be any cyclic ▱m Since ∠ X + ∠ Z = 2 rt ∠s (III 22), but ∠ X = ∠ Z (I 34), ∴ each = a rt ∠ So each of the ∠ s Y, P, is a rt ∠ Hence the ▱m is rectangular

585 The area of a square circumscribed about a ⊙, is double of the area of the inscribed square (See notes on IV 6, 7, General Notes) (*Bom Ex Pap*, *1865*)

586 The bisectors of the ∠s of a regular polygon, are concurrent (*See p 274, Ex 1, Text*)

587. Circumscribe a *rhombus* about a given ○. (*Cf. Bom Ex Pap*, *1865*)

Let O the centre of the given ○ Draw any two diameters AB, CD, and at their extremities, draw ⊥s PQ, RS, PS, QR forming by their intersection, the fig PQRS Then shall PQRS be a *rhombus* Since PQ, QR, RS and SP are the tangents to the ○ (III 16), ∴ the fig PQRS is circumscribed about the ⊙, and being a ▱m (I 28), is a *rhombus*

* **588** (*a*) In any △, the mid pts of the sides, the feet of the ⊥s from the vertices to the opposite sides, and the middle pts of the st lines joining the *orthocentre* to the vertices are *concyclic* (*See Ex. 32, p 281, Text*)

(*b*) If a polygon inscribed in a ○, is equilateral, it is also equiangular. (*See Text, p. 275*)

589 The ⊥s drawn from the centres of the three *escribed* ○s of a △, to the sides which they touch, are *concurrent*.

Let PQR be any Δ, and X, Y, R the centres of 3 *escribed* ⊙s Join XY, YZ, ZX forming the Δ XYZ, and about it describe a ⊙ (IV. 5), and let O be its centre Join XO and produce it to meet the ⊙ce in A, and cutting PQ in B. Then shall XO be $\perp$ to PQ. Join XR and produce it to meet the ⊙ce in C. Join AC Since $\angle$ XCA = a rt. $\angle$ (III 31), and $\angle$ XRY = a rt $\angle$; $\therefore$ ZY is ∥ CA (I 28) Hence arc YA = arc ZK (Ex 321), $\because$ $\angle$ YXA = $\angle$ZXC (III 27), and $\angle$ XQB = $\angle$ XZR ; $\therefore$ $\angle$ XBQ = $\angle$ XRZ (I 32) But $\angle$ XRZ is a rt $\angle$; $\therefore$ $\angle$ XBQ is a rt $\angle$. Hence XO or XB is $\perp$ to PQ So, if YO, ZO be joined, they shall be $\perp$ to QR, RP ; $\therefore$ the three $\perp$s are *concurrent*

590. Describe a ⊙ touching *three* given st lines (See Notes on IV 4, General Notes) *(Bom. Ex Pap, 1885.)*

591 If 4 ⊙s are described to touch every 3 sides of a quadrilateral, shew that their centres are *concyclic.*

Let PQRS be a quadrilateral Bisect $\angle$s P and Q by PO, QO meeting in O Then O shall be the centre of ⊙, touching the 3 sides SP, PQ, QR For, if $\perp$s OA, OB, OC be drawn to these sides, it may be proved that OB = OA = OC (I. 26). $\therefore$ the ⊙ with centre O and radius OA, will touch the three sides (III. 16) Let the bisectors of $\angle$s Q, R meet at D, and the bisectors of $\angle$s R,S meet at E, and of $\angle$s S, P meet at F Hence D,E,F are the centres of ⊙s touching 3 of the sides of the quadrilateral Then shall pts O, D, E, F be *concyclic.* Since, ext $\angle$DOF = ½ $\angle$P + ½ $\angle$Q (I 32), and the ext. $\angle$DEF = ½ ($\angle$R + $\angle$S), $\therefore$ $\angle$DOF + $\angle$DEF = ½ $\angle$s (P+Q+S) = 2 rt $\angle$s Hence, the pts O, D, E, F are *concyclic* (III 22, *Con*)

592 Inscribe (*a*) a ⊙, (*b*) a square, in a given *quadrant*

Let PQR be the given *quadrant*, PR being the arc

(*a*) Bisect rt. $\angle$PQR by QS, and draw SX $\perp$ to PQ Bisect $\angle$QSX by SY and draw YO ∥ to XS, meeting QS at O Then O shall be the centre of the reqd. ⊙. Draw OA $\perp$ to QR, $\because$ $\angle$ OYS = $\angle$YSX (I 29) = $\angle$YSO, $\therefore$ YO = OS (I. 6) Also OY = OA (I. 26), hence, the ⊙ with centre O and radius OS, passes thro' Y and A, and touches PQ, RQ, the radii of the quadrant at these pts (III 16) (since $\angle$s at Y and A are rt $\angle$s and touches the arc PR at S (III. 11), for O lies in SQ.

(*b*) The same const. being made, QXSB is the square reqd. From above, $\angle$ QSX = ½ a rt. $\angle$ (I 32), $\therefore$ $\angle$ SQY = $\angle$ QSX, and $\therefore$ QX = XS (I 6) But fig QS is a rectangle, $\therefore$ it is a square.

593 Three ⊙s, whose centres are P,Q,R, touch one another *externally two by two* X, Y, Z; shew that the

inriscbed ⊙ of the $\triangle$ PQR, is the circumscribed ○ of the $\triangle$ XYZ

Now PQ, QR, RP pass thro' Z, X, Y (III. 12), ∵ the common tangents at Z, X, Y meet at a pt O, and are equal, O is the centre of the ○ about $\triangle$ YXZ Join OP, OQ, OR Since O is the pt of the intersection of the tangents at Y and Z, ∴ OP bisects $\angle$P (III 17, Cor) So, OQ and OR are the bisectors of $\angle$s at Q and R, OZ, OY, OX are $\perp$s to the sides of $\triangle$ PQR (III 18), ∴ the ○ XYZ is inscribed in $\triangle$ PQR

594 *With the aid of an isosceles $\triangle$, such that each of the $\angle$s at its base is 7 times the $\angle$ at the vertex, to inscribe a regular* quindecagon *in a given* ○ (Cal. U Pap, 1886)

In the given ○, whose centre is O, inscribe a $\triangle$ QPR equiangular to the given isosceles $\triangle$, so that the vertical $\angle$ P $=\frac{1}{7}$th of the base $\angle$s at Q and R (IV 2) Then QR shall be a side of a regular *quindecagon* inscribed in the ○ Join QO, RO, ∵ $\angle$ P + $\angle$ Q + $\angle$QRP = 2 rt $\angle$s, ∵ $\angle$P $=\frac{1}{15}$ of 2 rt $\angle$s Now, $\angle$ QOR = 2 $\angle$ P (III 20) $=\frac{2}{15}$ of 2 rt. $\angle$s $=\frac{1}{15}$ of 1 rt $\angle$, ∴ arc QR $=\frac{1}{15}$ of the whole ○ce, and thus QR is a side of a regular *quindecagon* inscribed in the ○

595 Shew how regular polygons of 16 and 20 sides respectively, can be inscribed in a given ○ (Cal F A Pap, 1866)

(i) Let O be the centre of the given ○ Draw any two radii OR, OS, at rt $\angle$s to each other Bisect $\angle$ ROS by OP, meeting the ○ce at P, and bisect $\angle$ ROP by OX, meeting the ○ce at X. Join RX. Then RX is the side of a regular fig of 16 sides inscribed in the ○ Since $\angle$ ROS is a rt $\angle$, ∴ $\angle$ ROX is $\frac{1}{4}$ of a rt. $\angle=\frac{1}{16}$ of 4 rt $\angle$s Hence, the arc RX $=\frac{1}{16}$th of the whole ○ce, and thus RX is a side of a regular polygon of 16 sides inscribed in the ○

(ii) Let SA be a side of a regular *pentagon* inscribed in the ○, (IV 11) Join OS, OA Bisect $\angle$ SOA by OB meeting the ○ce at B, and bisect $\angle$ SOB by OC meeting the ○ce in C Join SC Then SC is the side of a regular polygon of 20 sides inscribed in the ○. Since $\angle$ SOA $=\frac{1}{5}$ of a rt $\angle$ ∴ $\angle$SOC $=\frac{1}{20}$th of 4 rt $\angle$s Hence, the arc SC $=\frac{1}{20}$th of whole ○ce and thus SC is a side of a regular polygon of 20 sides inscribed in the ○.

*596 Prove that —(a) the centre of the nine-pts. ○, is the middle pt of the st. line which joins the orthocentre to the circumscribed centre, (b) the radius of the nine-pts ○, is $=\frac{1}{2}$ the radius of the circums-

cribed ⊙; (c) the centroid is collinear with the circumscribed centre, the nine-pts centre, and the orthocentre. *(See Ex 33 p 282, Text)*

597. *Each diagonal of a regular pentagon is ∥ to the sides with which it is not* conterminous (See Notes on IV 11, General Notes)

598 *If the vertical ∠ of a △ be acute, and from the extremities of the base, two st lines be drawn, making with the sides, ∠s = to the ∠s at the base, these two st lines shall form with the base another △, of which the exterior ∠ at the vertex =* twice *the ∠ of the first △.*

Shew what changes will take place in the problem, if the vertical ∠ be (i) a rt ∠, (ii) an obtuse ∠ (Cal F A. Pap, 1868)

Let PQR be the △, of which the vert ∠QPR is *acute* Draw QX and RY making ∠s PQX and PRY = ∠s PQR and PRQ respectively. Since ∠ P is acute, ∴ ∠PQR + ∠PRQ > a rt ∠ (I 32) Thus (∠XQR + ∠YRQ) > 2 rt ∠s Hence, XQ, YR when produced, will meet Let them meet at Z Produce QZ to O Then shall ∠RZO be = 2 ∠P Since, ∠XQR + ∠YRQ + ∠RZO = 4 rt ∠s (I 32, Cor 2), and 2 (∠PQR + ∠PRQ + ∠QPR) = 4 rt ∠s; ∴ ∠XQR + ∠YRQ + ∠RZO = 2 (∠PQR + ∠PRQ + ∠QPR) But ∠XBR = 2 ∠PQR, and ∠YRQ = 2 ∠PRQ, ∴ ∠RZO = 2 ∠QPR

(i) When the ∠at P is *a rt* ∠, (∠XQR + ∠YRQ) will be 2 rt ∠s Thus XY will be ∥ to YR

(ii) When the ∠ at P *is obtuse*, the st lines XQ and YR will meet on the same side of QR, in which P is, and then, 2 ∠P will be = 4 rt ∠s = ∠YRZ

599 *In a given ○, inscribe a △, such that two of its sides may pass through two given points, and the third side be of* given length.

Let PQR be the given ○, X, Y, the given pts *(both within* the ○), and AB the *third* side of given length, placed in the ○ Join XY, and on it, describe a segment of a ○, containing an ∠ = the ∠ in the segment BPA (III 33), and cutting the given ⊙ at P Join PX, PY, and produce them to meet the ○ce at QR Join QR Then PQR is the reqd △, ∵ ∠QPR in segment XPY = ∠ in segment BPA (cons), ∴ arc QR = arc AB, (III 26) Hence, chord QR = chord AB (III 29)

600 If a ○ be inscribed in any △, the points of contact shall divide the sides, into segments, such that any one side together with the remote segment of either of the other two sides, shall be = ½ the sum of the three sides (Cal U. Pap, 1865)

Let the ◯XYZ be inscribed in the ΔPQR, to touch the sides PQ, QR, RP at Y, Z, X respectively Then shall $(QR+PY)=\frac{1}{2}$ the sum of the sides of the Δ PQR Since $QZ=QY$ and $RZ=RX$; $\therefore QR=QY+RX$. Also, $PX=PY$, hence, $QR+PY=QY+RX+PX=QY+PR$ Thus, $QR+PY=\frac{1}{2}(QR+PY+QY+PR)=\frac{1}{2}(QR+PQ+PR)$

APPENDIX.

MADRAS ENTRANCE PAPERS, (SOLUTIONS)

Exercises 601-632

1857.

5 ABCD is a quadrilateral figure, the sides AB, DC are produced to meet in E and the sides AD, BC to meet in F. Then that the $\angle$ BCD = sum of the angles at A, E and F

$\angle BCD=\angle CBE+CEB$
Alson $\angle CBE=\angle FAB+AFB$
$\therefore \angle BCD=\angle FAB+\angle AFB+\angle CEB$

1858.

6 In a circle, whose radius is "*a*", find the length of a chord, whose shortest distance from the centre is "b"

$\left.\begin{array}{l}OA=\text{"a"}\\ OC=\text{'b'}\end{array}\right\} AC^2=OA^2-OC^2$

$\therefore AC^2=a^2-b^2$, $\therefore AC=\sqrt{a^2-b^2}$
But $AB=2AC$, $AB=2\sqrt{a^2-b^2}$

1860

10 If $AC=\frac{1}{n}$ *th* of AB, shew that the diagonal of the square described upon AB is equal to $\sqrt{2}\ n\,AC$.

$AD^2=2AB^2$
But $AB=n^2\,AC$
$\therefore AB^2=n^2AC^2$
$\therefore AD^2=2n^2AC^2$, $\therefore AD=\sqrt{^2}\ nAC$.

11. If any number of concentric circles be cut by another circle, the common chord shall be parallel

In the Δs ACA′, ADA′

AC=AD, A′C=A′D, AA′ is common,

$\therefore \angle CAO = \angle DAO$.

In the Δs ACO, ADO

AC=AD, AO is common,

$\angle CAO = \angle DAO$, $\therefore \angle AOC = \angle AOD = 1$ rt $\angle$

$\therefore$ AA′ cuts CD at rt $\angle$s

Similarly, it can be shewn that it cuts all other common chord at right angles Hence they are all parallel

1861.

6 Two triangles stand upon a common base, either on the same or opposite sides, draw a line through the vertices and cutting the base, produced if necessary If the distance measured along this line, of the one vertex from the base, be double that of the other; the area of the 1st triangle, is also double that of the second Δ

Let ACB, AC′B be the Δs

C′D=2CD, $\therefore$ CC′=CD

$\therefore$ AC′D=2 ACD

or AC′B+C′BD=2 ACB+2 CBD——(1)

But C′D=2 CD, $\because$ C′BD=2 CBD

$\therefore$ A′CB=2ACB from (1)

12 If two tangents to a circle, be drawn from the points A and B on the circumference, and intersect at C, and if AC be produced to meet the radius through B, at D; shew that the rectangle contained by AC and AB=the rectangle contained by BD and the radius.

$AD^2 = DB \ DP$

$= DB^2 + DB \ BP$

$= DB^2 + 2OB \ DB$

Also $AD^2 = AC^2 + CD^2 + 2AC \ CD$

$\therefore AC^2 + CD^2 + 2AC \ CD = DB^2 + 2OB \ DB$

Add AC^2

$\therefore 2AC^2 + CD^2 + 2AC \ CD = AC^2 + DB^2 + 2OB \ DB$

But AC=CB

$\therefore AC^2 + DB^2 = CB^2 + DB^2 = CD^2$

$\therefore 2AC^2 + CD^2 + 2AC \ CD = CD^2 + 2 OB.DB$

or $AC^2 + AC \ CD = OB.DB$

or $AC \ AD = OB.DB$

1862

7 From AB, the side of an isosceles Δ (or from AB produced), cut off AD=AC the base, and from AC produced (or from AC) cut off AE=AB Let BC and DE meet in F Prove that DF=CF, and also that BF=EF Show also that DE and BC cannot be bisected in F

In the Δs ABC, ADE
AB=AF, AC=AD
The contained $\angle$ is common
$\therefore$ $\angle$ABC= $\angle$AED
BC=DE

In the Δs BDF and CEF, $\angle$CEF= $\angle$DBF, $\angle$CFE= $\angle$DFB, $\therefore$ $\angle$FCE= $\angle$BDF,

Also DB=CE
$\therefore$ the Δs are equal
i e, CF=FD, BF=FE
We see that DE=BC

Hence, if they bisect each other, and the extreme point be joined, the figure will be a parallelogram, $\therefore$ BD shall be parallel to CE or AE, which is absurd

10 If in Δ ABC, the side CA exceed BC, by half BA, shew that the square on CA is greater than the square on BC by the rectangle contained by BC and BA, together with the square on half BA

$$CA=BC+\tfrac{1}{2}\ BA$$
$$CA^2=BC^2\ (\tfrac{1}{2}\ BA)^2+BC\ BA\ (II\ \ 4)$$

1863

3 *(b)* The sum of the diagonals of a parallelogram, is less than the sum of the four lines, which are drawn from any point, to the angles of the parallelogram

$$OA+OC>AC,\ OB+OD>BD$$
$$\therefore\ OA+OB+OC+OD>AC+BD$$

12 Two circles of unequal radii, touch each other externally, and any st line is drawn through the point of contact cutting both circles, shew that the diameter drawn through the points of intersection, are parallel

Join OO'; $\therefore$ it will pass through A
$\angle$ OAC= $\angle$O'AB
But $\angle$OCA= $\angle$OAC and $\angle$O'AB= $\angle$O'BA
$\angle$OAC= $\angle$OCA= $\angle$O'AB= $\angle$O'BA
$\therefore$ OC and O'B are parallel

1864.

Part I.

4 On one of the sides of a Δ, describe an isosceles Δ equal to the original Δ.

Let ABC be the triangle

Through C draw a line parallel to AB, bisect AB in D and draw DE perpendicular to AB, meeting the parallel line at E

Join AE and BE

In the Δs ADE and BDE,

$AD = BD$, ED is common

$\angle ADE = \angle BDE$, being each a rt $\angle$

$\therefore$ $AE = EB$, ΔAEB is isosceles and it is $= \Delta ACB$.

Part II

6 Two equal circles cut one another in A and B The straight line joining their centres, cuts the circles in C and D Shew that ABCD is a rhombus Can this rhombus, under any circumstance, become a square ?

Since $AC = AD = BD = BG$

$\therefore$ ABCD is a rhombus

1865

4 ABC is a given Δ. Describe a Δ $PQR =$ to BC having a side $PQ - AB$ and a side $PR = 2\ AC$, show that two such Δs may be described

Through C draw a line parallel to AB, and with the centre A, and radius = to 2AC, describe a $\smile$ cutting the parallel line at M Join BM

The Δ $ABM = \Delta$ ABC

It has one side AB and the other $AM = 2\ AC$

$\therefore$ Δ PQR is = to Δ ABM

7 ABC is an acute angled triangle. From the points A, B, C perpendiculars AD, BE, CF are drawn upon the opposite sides BC, CA, AB, shew that the squares on the three sides of the Δ is double the sum of the rectangles contained by AB AF by BC BD and by AC AF.

We have

$AC^2 = AB^2 + BC^2 - 2\ BD.BC$

$AB^2 = BC^2 + AC^2 - 2\ CE\ CA$

$BC^2 = AC^2 + AB^2 - 2\ AF.AB$

Adding and cancelling, we have

$AB^2 + AC^2 + BC^2 = 2\ (AF\ AB + BD.BC + CE\ CA)$.

8 By hypothesis, AC is the diameter of the smaller ○

∴ ∠ADC = 1 rt ∠ (III 21)

∴ AB is bisected at D (III. 3).

11 If two circles touch each other internally, the radius of the larger being double that of the smaller, prove that any chord of the larger circle passing, through the point of contact, is bisected by the circumference of the smaller

Let AC C′M be the diameter of the larger circle

∴ C′ the centre of the larger ○ lies on the circumference of smaller one, draw any line APM through the point of contact

Join C′P and MM′

Since ∠AMM′ = ∠APC′ = 1 rt ∠ (III 31)

∴ MM′ and C′P are parallel

But C′P bisects AM′, ∴ it bisect AM at P

MADRAS MATRICULATION PAPERS, (SOLUTIONS)

1866

3 $QR = \frac{1}{3} AB$ $BX = \frac{1}{3} AB$

∴ $QR = BX$, ∴ $QX = BR$, ∴ $DE = BR$

Now, in the Δs CDE, BGR

∠CDE = ∠A = ∠B, ∠E = ∠R and DE = BR

∴ DC = BG, ∴ AD = CG

5 Take CD = Z i e, the given difference

On CD, describe a segment containing an ∠ = ½ (∠P − ∠R).

At C, make ∠DCA = ∠R

Produce CD to B, and make ∠DAB = ∠ADB

Then ABC is the reqd triangle.

Now ∠ABD = ∠ADB (cons)

= ∠DBC + ∠C (I 32)

∠ABC = 2 ∠DBC + ∠C

= ∠P − ∠R + ∠R = ∠P

∴ ∠BAC = ∠Q

Also CD = the given *difference*

9 $CD^2 = CB^2 + BD^2$

$= CB^2 + BD\ BE$ (∵ $BD = BE$), (III 3)

$= CB^2 + CB\ BA$ (III 34)

$= CB\ CA$

$= CI^2$ (III 36)

∴ $CD = CI$.

10. $\angle ABC = \angle ACB = \angle CBD$
$\angle ACB = \angle CDB$
$\therefore \angle BAC = \angle BCD$
$\therefore \triangle$s ABC, BDC are similar (VI 4)
$\therefore \frac{AB}{BC} = \frac{BC}{BD}$, $\therefore AB \cdot BD = BC^2$

1867.

3 $\angle E = \angle F$ (I 29)
But $\angle E = \angle ABE = \angle FBG$
$\therefore \angle BFG = \angle FBG$, $\therefore BG = GF$,
i.e. BGFC is a rhombus,
$\therefore$ BF, CG bisect each other at rt angles
5 $CD^2 = ED^2 + EC^2$ (I. 47)
$= (AC+BD)^2 + (AC-BD)^2$
$= 2AC^2 + 2BD^2$.
10. KCG is one st line
$CD^2 = CO^2 - OD^2$
$= (DH+OA)^2 - OD^2$
$= (DH+OA+OD)(DH+OA-OD)$
$= AH \cdot BG$

1868.

3 Make $AD = AB$
$AB + BC > AC$
$> AD + DC$
$\therefore BC > DC$.
4 $AC > AD + DC$
$AB + AC > 2AD + 2DC$
$> BC + AE + BD + CE$
10 $\angle P + \angle B = 2$ rt angles
B M, P, C are concyclic,
$\therefore AM \cdot AB = AP \cdot BC$ (III 36)
11. $\angle DCE = \angle QPA = \angle QBA$, $\therefore$ CD ‖ to AB.

1871.

3. $BD = CD$, $\therefore \angle DBC = \angle DCB$
$\therefore 2\angle DBC = 2\angle DCB$
i.e. $\angle ABC = \angle ACB$; $\therefore AB = AC$
5 $\angle BAD = 90° - \angle B$, $\angle CAD = 90° - \angle C$
$\therefore \angle BAD \sim \angle CAD$
$= \angle B \sim \angle C$
or $2\angle DAE = \angle B - \angle C$

7. $AB^2 = AC^2 + CB^2 - 2\,AC\,CD$ (II 13)
But $AB = AC$, $CB^2 = 2\,AC\,CD$

9 Join C, C′
The line of centre, must pass through A, the point of contact
$\therefore \angle C = \angle C'$
Join PA, QA
Now $\angle C + \angle CPA + \angle CAP = \angle C' + \angle CAQ + \angle C'QA$
or $\angle C + 2\angle CAP = \angle C' + 2\ \angle C'AQ$
But $\angle C = \angle C$
$\angle CAP = \angle C'AQ$
$\therefore$ AP,AQ form one st line

PUNJAB ENTRANCE PAPERS, (SOLUTIONS)

Exercises 633-668

1876

9 Shew that the difference of the base and $\angle$s of any Δ, is double the $\angle$ contained by a line drawn from the vertex perpendicular to the base, and another bisecting the $\angle$ at the vertex

Let AD be the perpendicular and AE the bisector of the vertical angle

$\angle ABC = \angle ADC - \angle BAD$
$\angle ACB = \angle ADB - \angle CAD$
$\therefore \angle ABC - \angle ACB = \angle CAD = \angle BAD$
But $\angle CAD = \angle CAE + \angle DAE$
$= \angle BAE + \angle DAE = \angle BAD + 2\ \angle DAE$
$\angle ABC - \angle ACB = \angle BAD + 2\ \angle DAE - \angle BAD = 2\ \angle DAE$

1877

6 In a given straight line, find a point equidistant from two other points without the line

Let AB be the st line and C, D the points,

Join CD and bisect it at rt. $\angle$s by EF, meeting AB in F. Join CF and DF

In the Δs CEF and DEF,

CE=DE, EF is common, and the contained angles are equal, being each=a rt. $\angle$, CF=DF

10 Draw through a given point, a st line which shall make equal $\angle$s with two other given st lines

Let the st line produced, meet A.

Let D be the point Bisect the $\angle$ BAC and draw DE perpendicular to the bisector, produce DE both ways to meet the given line at F,G

In the Δs AEF and HEG

The side AE is common, $\angle$FAE $=$ $\angle$ GAE

$\angle$ AEF $=$ $\angle$ AEG, being each $=$ a rt $\angle$.

$\therefore$ $\angle$ AFE $=$ $\angle$ AGE

1878

2 (*b*) If P be a point in a side AB of a parallelogram ABCD, and PC, PD be joined, the Δ PAD $+$ ΔPBC $=$ Δ PDC

Through P draw PQ parallel to AD or BC

$\because$ APQD, PBCQ are both parallelograms,

$\therefore$ ΔAPD $=$ Δ PDQ, Δ BPC $=$ ΔPCQ

$\therefore$ Δ APD $+$ Δ BPC $=$ Δ PDQ $+$ Δ PCQ $=$ Δ PDC

6 Given the base of a Δ, the vertical $\angle$ and the length of the line drawn from the vertex to the middle point of the base, describe the Δ

On AB describe a segment, containing an $\angle$ $=$ to the given $\angle$ Let D be the middle point of AB

With D as centre and radius $=$ to the given median line, describe a $\bigcirc$ cutting the segment at C, then ACB is the Δ required.

$\because$ $\angle$ ACB $=$ the given angle

CD $=$ the given median

9 Prove that the point of intersection of the diagonals of a given square, described on the hypotenuse of a rt $\angle$d Δ, is equidistant from the two sides containing the rt. $\angle$.

$\angle$ EAP $+$ $\angle$ CAB $=$ 1 rt. $\angle$ $=$ $\angle$CAB $+$ $\angle$ABC

$\therefore$ $\angle$ EAP $=$ $\angle$ABC.

Also $\angle$ OAE $=$ $\angle$ OBA, each being half a rt. $\angle$;

$\because$ $\angle$ OAP $=$ $\angle$ OBQ; also $\angle$APO $=$ $\angle$BQO.

$\therefore$ $\angle$AOP $=$ $\angle$ BOQ

Now in the Δs OAP and OBQ; OA $=$ OP ($\because$ each being half the diagonal of the square), and the adjacent $\angle$s are equal, $\therefore$ OP $=$ OQ

1879

8 In a given $\odot$, to inscribe four equal $\odot$s touching each other, and the given $\odot$.

Draw any two diameters at rt $\angle$s to each other; take any quadrant AOB, bisect the $\angle$AOB by CO

At C, make $\angle$ OCD $=\frac{1}{4}$ rt. $\angle$ and at O, draw DE at rt $\angle$s to OB

Now $\angle$ OCD $=\frac{1}{4}$ rt $\angle$, $\angle$COD $=\frac{1}{2}$ a rt $\angle$
$\therefore$ $\angle$ CDO $=\frac{3}{4}$ rt. $\angle$
Also $\angle$ ODE $=1$ rt $\angle$, $\angle$ EDC $=\frac{1}{4}$ rt $\angle$
$\angle$ EDC $=\angle$ ECD, $\therefore$ EC $=$ ED
But, it is easy to see that, ED $=$ EF
$\therefore$ EC $=$ ED $=$ EF
$\therefore$ a $\bigcirc$ with E as centre and EC as radius will touch OA and OB at F and D

From geometry, we can describe the other $\bigcirc$s also

1881

5 Inscribe an equilateral Δ in a $\bigcirc$, and compare its area with that of a regular hexagon, inscribed in the same circle.

Let A be a point on the $\bigcirc$ce of the $\bigcirc$ At A draw two lines AB, AC making an $\angle$ of 120 degrees with each other

Join BC, bisect it at D, and draw DE at rt $\angle$s, meeting the $\bigcirc$ce at E Join BE and CE

Since $\angle$ BAC $+$ $\angle$ BEC $=180$ degrees

But $\angle$ BAC $=120$ degrees, $\angle$ BEC $=60$ degrees

Also it can easily be shewn that BE $=$ EC, $\therefore$ $\angle$BCE $=$ $\angle$EDC
But $\angle$ BCE $+$ $\angle$EBC $=$ 120 degrees, $\therefore$ each $=60$ degrees
$\therefore$ Δ BCE is equilateral

6 Two $\bigcirc$s touch internally at A A straight line touches one $\bigcirc$ at P and cuts the other at Q and R Prove that PQ and PR subtend equal $\angle$s at A

Let TAT′ be the common tangent to the two $\bigcirc$s, and let the lines AR, QR meet the smaller $\bigcirc$ at M, P

$\angle$ MAT $=$ $\angle$ MPA $=$ $\angle$PQA
$\angle$ APQ $=$ $\angle$AMP
In the Δs APQ, APM, two $\angle$s are equal
$\therefore$ $\angle$ PAQ $=$ $\angle$ PAM

1883

3 If AB, BC be equal sides of an isosceles Δ, and a $\bigcirc$ with centre B and distance BA cut AC (or AC produced) in E; and BF be taken in AB (or AB produced, if E lies in AC produced) $=$ to CE, prove that $\angle$ CFA is $=$ $\angle$FAC

Join BE and CF
It is easy to see that, ΔABE $=$ ΔACF and isosceles
$\therefore$ $\angle$CAF $=$ $\angle$CFA

6 If two straight lines be drawn through any point on a diagonal of a square, parallel to the sides of the square; the points where these lines meet the sides, will be on the ⊙ce of a ◯, whose centre is at the intersection of the diagonals

Let LOMP be a square, ∴ LO=OM, OP is common to Δs POM, POL, the contained $\angle$s are equal, ∵ LP=PM, so PR=PT

Also the perpendicular from P bisects RM and AD

∴ Δ PRM is isosceles and PR=PM

∴ PM=PL=PR=PT

i e L, M, T, R lie on the circumference of the ◯.

1885.

2 (*b*). $\angle$ BCD + $\angle$CDB + $\angle$DBC = 2 rt $\angle$s

$\angle$HBA + $\angle$KCA = 2 rt $\angle$s + $\angle$BAC

∴ $\angle$EBH + $\angle$FCK = one rt $\angle$ + ½ $\angle$BAC

or $\angle$DBC + $\angle$BCD = 1 rt $\angle$ + ½ $\angle$BAC

or $\angle DBC + \angle BCD - \frac{\angle BAC}{2} = 1$ rt $\angle$

Subtracting (2) from (1) we have

$\angle BDC + \frac{\angle BAC}{2} = 1$ rt $\angle$

3 (*b*) From D draw DC at rt $\angle$s to the tangent, ∴ DC passes through the centre (III 19), and is also $\perp$ to AB

Now $\angle$AEC = $\angle$BEC, being rt $\angle$s,

$\angle$CAE = $\angle$CBE (I 5)

∴ $\angle$ ACD = $\angle$BCD (I 32)

∴ the arc AD = the arc BD (III 26)

7 All parallelograms about which a ⊙ can be described are rectangular, for then the opposite $\angle$s become = to 2 rt $\angle$s, and these $\angle$s are equal

But, rhombus is not rectangular, ∴ rhombus is not a cyclic fig.

1886.

3 (*b*) Bisect AB in D Draw BE $\perp$ to AB, making BE = a side of the given sq. (K) Join DE From centre D with radius DE, describe a ⊙ cutting AB produced in C

Now AC CB = $CD^2 - BD^2$ [II. 6.

$= DE^2 - BD^2 = BE^2 = K^2$

1887.

5 (*b*) Make $\angle$MON=Supplt of $\angle$A
∴ $\angle$MOS=$\angle$B
Draw tangents at M or S
PQR is the Δ
$\angle$P=Supplt of $\angle$MON=$\angle$A
$\angle$Q=$\angle$MOS=$\angle$B

8 $\angle$PQD=$\angle$PRD
=Complt of $\angle$PRS=$\angle$PSR
Since $\angle$SRR= a rt. $\angle$,
∴ T,Q, P,S are concyclic
$\angle$TPS=$\angle$TQS.

1888

8. R and r be the radii of circumscribed and inscribed ◯s.
Bisect $\angle$A and at a distance r from AB, draw a $\perp$ meeting the bisector in S With distance=s, and radius=R^2-2Rr

See Ex 38, (1) p 283, Text.

Describe a ◯ with centre A and radius=R intersecting in I with centre I and radius=R, describe a ◯ intersecting AB in B.
Through B, draw a tangent BC to the ◯ ABC is the Δ required

6 (*b*) Draw GF bisecting AD at rt $\angle$s in H Then the centre of ◯ CAD lies on GH, so also the circum-centre of ΔABD Call them F and G

Join FA, FD, AG, GD.
Now $\angle$AFD=(reflex angle)=2 $\angle$ACD
$\angle$AFD=2 $\angle$BCD
∴ $\angle$AFH=$\angle$BCD
Again $\angle$AGF=$\frac{1}{2}$ $\angle$ AGD=$\angle$ABD=$\angle$BCD
∴ $\angle$ AFH=$\angle$AGF, AG=AF
∴ the ◯ with centr F=the ◯ with centr G

1889

1 The area of the ▭m. AC=2 Δ ABC

$$=2\frac{AE\ BC}{2}$$
$$=AE\ BC$$
=the product of the base and height.

2 Bisect AC in D Join BD
$\angle$ACB=2 $\angle$BAC=$\frac{2}{3}$ of one rt $\angle$
But we know that, BD=$\frac{1}{2}$ hypotenuse=DC ,
∴ $\angle$DBC=$\angle$BCD=$\frac{2}{3}$ of one rt $\angle$

$\therefore \angle BDC = \frac{2}{3}$ rt $\angle$.

$BC = CD = \frac{1}{2}$ hypotenuse.

4 $AC^2 = AB^2 + BC^2 - 2\,BC\,BD$. [II. 13.

Let $AB = c$, $AC = b$, $BC = a$.

Then $b^2 = c^2 + a^2 - 2a.\ BD$

$$\therefore BD = \frac{a^2 + c^2 - b^2}{2a}$$

$$\therefore AD = \sqrt{c^2 - \frac{(a^2 + c^2 - b^2)^2}{4a^2}}$$

$$= \frac{1}{2a}\sqrt{(2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)}$$

$$= \frac{1}{2a}\sqrt{(a+b+c)(a-b+c)(c+b-a)(b+a-c)}$$

(let $2s = a + b + c$), then

$$AD = \frac{1}{2a}\sqrt{2s(2s-2b)(2s-2a)(2s-2c)}$$

$$= \frac{4}{2a}\sqrt{s(s-a)(s-b)(s-c)}$$

But we know that, the area of Δ $ABC = \frac{a}{2}.\ AD$

$$= \frac{a}{2}.\ \frac{4}{2a}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

When $\angle B =$ a rt $\angle$,

AD coincides with AB, and the area $= \frac{AB.BC}{2} = \frac{ac}{2}$.

5. (*b*) Take any two points A, B on the circumference;
Join AB; make $\angle BAC = \frac{1}{3}$ of 2 rt. $\angle$s
Join BC Take any point D in the arc BC.
Join BD, CD, $\angle A + \angle D = 2$ rt. $\angle$s.
$\therefore \angle BDC = \frac{2}{3}$ of 2 rt. $\angle s = 2\ \angle A$ [III. 22.

1891.

The angle of a hexagon $= \frac{4}{3}$ of 1 rt. $\angle$.

Join A, D, any two points on the ⊙ce. Make $\angle ADC = \frac{2}{3}$ of a rt. $\angle$ (an angle of an equilateral Δ).

Join AC. Take any point B on the arc ABC.

Then $\angle ABC = 2$ rt. $\angle s - \angle BDC = \frac{4}{3}$ of a rt. $\angle$.

6. Draw FCG parallel to DF. Now $\Delta ACF = \Delta GCB$, for

CB = CA, ∠BCG = ∠ACF, ∠CAF = ∠CBG (I 26) To each, add the fig ACG TD, then AB TD = FG TD = DF CE = ½DF AB

7 Δ SRP is a rt ∠d Δ,
∠QQR = ∠PBA [III 33
= 90 deg − ∠PAB = ∠ARO = ∠QRP
∴ QR = QP, SQ = QR

8 AD is greatest, when AE is greatest,
Now, A is a *fixed* point, and the point E is *variable*
Join E to the centre O
Then ∠s at E, are rt. ∠s
∴ the locus of E is the ◯, on OB as diameter

Now the greatest st line that can be drawn from A to the ◯ce, is that, which passes through the centre O′

the diagonal is greatest, when it passes through O′

6 Produce BA to F, making AF = AB or AC,
BF = AB + AC To shew AB + AC < BD + DC
∠FCB = ∠ angle
∠FED = ∠DEC, both being rt angles
In Δs FDE and CDE, ∠FED = ∠DEC
FE = EC, and ED is common,
∴ FD = DC
From Δ FDB, we have FD + DB < FB i e < AB + AC
∴ isosceles Δ has the *least* perimeter

7 ∠s at E and F are rt ∠s
∴ A, F, H, E are concyclic.

Again F and E lie on the semi ◯ on BC as diameter, (G being centre)

$$\begin{aligned}
\therefore\ & AB\ BF + AC\ CE \\
&= BH\ BE + CH.\ CF \qquad \text{[III 36} \\
&= BH^2 + CH^2 + BH\ HE + CH\ HF \\
&= HG^2 + GB^2 - 2\ BG\ GD + HG^2 + GC^2 + 2\ CG\ GD \\
&\qquad + 2\ CH\ HF \\
&= 2r^2 + 2\ HG^2 + 2\,(r^2 - GH^2) \\
&= 4r^2 = (2r)^2 = BC^2
\end{aligned}$$

1892.

3 In the Δs ABD and ACE,
AB = AC, ∠A is common, ∠ABD = ∠ACE, ∴ ΔAEC = ΔADB

From the whole Δ ABC, subtract each of the equal Δs AEC, ADB, then the remainders are equal,

i e $\triangle$ BEC = $\triangle$ BDC
$\therefore$ BC is parallel to ED. [I. 39

8 $\angle$BDC = $\angle$ABC = a constant,
and $\angle$A is constant,
$\therefore$ $\angle$BDC + $\angle$A = a constant,
$\therefore$ $\angle$ABD + $\angle$ACD = 4 rt $\angle$s − ($\angle$BDC + $\angle$A) = constant.

9 F, B, D, O are concyclic, [Conv III 22
So also O, D, C E are concyclic
$\therefore$ $\angle$BDF = $\angle$BOF = $\angle$COE = $\angle$CDE
$\therefore$ DF and DE are equally inclined to AD

1893

7 Inscribe a pentagon in the ☉, [IV 11
Let AB be one of its sides,
Bisect the arc AD at C
Then each of the arcs AC, BC = $\frac{1}{10}$ part of the ☉ce Join AC, CB, and place successive chords equal to them

8 Since O is the in-centre of $\triangle$ EDF
$\therefore$ OE, OF, OD bisect $\angle$s E, F, D, respectively,
$\therefore$ the sum of half the angles E, F D = one rt $\angle$,
Now $\angle$ AGC = $\angle$ BGD = 2 rt $\angle$s − ($\angle$ GBD + $\angle$ GDB)
= 2 rt. $\angle$s − ($\angle$ GBD + $\angle$ BDF + $\angle$ GDF)
= 2 rt $\angle$s − ($\frac{1}{2}$ F + $\frac{1}{2}$ E + $\frac{1}{2}$ D)
= 2 rt $\angle$s − 1 rt $\angle$
= One rt $\angle$.
$\therefore$ AG is $\perp$ to BC
So, it may be shewn that, CF is $\perp$ to AB, and so on

1894.

3 DH and DU are respectively parallel and half of AC and AB, $\therefore$ DH = DU, and $\angle$HDB = $\angle$UDC

$\therefore$ $\angle$HDE = $\angle$UDE, and DE is common,
$\angle$DEF = $\angle$DEG [I 4

Again, in the $\triangle$s EOF, EDG, $\angle$s at D are rt $\angle$s,
$\angle$DEF = $\angle$DEG, DE is common, $\therefore$ DF = DG,
$\therefore$ the diagonals AE, FG bisect each other at rt $\angle$s,
$\therefore$ the fig AFEG is a rhombus.

1895

6 (a) AB^2 = AD AE, AC^2 = AD AE, [III. 36
$\therefore$ AB = AC

20 Let AB be one of the given sides Draw BC at rt $\angle$s to AB, from centre B with radius to the given perp K describe a ☉

From A draw ADC a tangent to it, then ABC is the Δ reqd. BD is prep to AC [III. 18

The cons fails, where AB is not > than the given length Q

ALLAHABAD ENTRANCE PAPERS, (SOLUTIONS).

Exercises 669-676.

1889.

2 Make ∠BAD=the given ∠, and AD=Q

Join DB Make ∠DBC=∠BDC

Now Δ ACB is that reqd

AC+CB=AC+CD=AD=K.

4 See Text book, Page 111, Ex 21

7 (*b*) On AB, describe a segment ACB containing ∠= to ∠E Then the vertices of the Δs drawn on AB, must lie on the arc ACB [See III Con of 21

Bisect AB at D and draw DC ⊥ to it Then the Δ ACB is isosceles The areas of all Δs on AB, *vary* as half their altitudes, since AB is constant

But CD is the greatest altitude, since it is the ⊥ distance between the tangent at C and AB,

∴ the *area* of the isosceles Δ, is the *greatest*

1890

10 (*b*) On CB, describe the isosceles Δ CAB having each of the ∠s at the base, double of the third ∠

Then ∠BCA=$\frac{4}{5}$ of 2 rt ∠s

Bisect it, by CD, then ∠BCD=$\frac{4}{5}$ of a rt ∠

1892

2 (*b*) Take any other point E Join EA, EB, EC, ED

EA+EC>AC, EB+ED>BD

. EA+EB+EC+ED>FA+FB+FC+FD,

F is the point, the sum of whose distances from the angular points is the least

1893

7 Let A, B, C be the three points, not in the same st line Join AB, BC AC,

Now describe a ◯ about ΔABC [IV 5

1894.

8 Join AD, $\angle AEC = 2\angle ADC$; $\angle DEB = 2\angle DAB$
$\therefore \angle AEC + \angle DEB = 2\angle AEC$

9 The given $\Delta = \Delta\ AiB + \Delta\ AiC + \Delta\ BiC$
$= \frac{AB+BC+CA}{2} \cdot i$

$\therefore \frac{\Delta}{S} = i$ [where S=½ the sum of the sides

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BOMBAY MATRICULATION PAPERS, (SOLUTIONS).

Exercises 677-692

1886 87.

4. since Δs PAQ, PBQ are equal, and on the same base,
$\therefore$ AD=BE
$\because$ $\Delta\ ACD = \Delta\ ECB$, in all respects
$\therefore$ AC=BC. [I. 26

8 The diameter ACB passes through A and B, since DA and EB are parallel, [III. 17 cor
$\angle DCF = \frac{1}{2}\angle ACF$; $\angle ECF = \frac{1}{2}\angle BCF$
$\therefore$ $\angle DCE = \frac{1}{2}$ of 2 rt $\angle$s = 1 rt. $\angle$.

6 $ED^2 = GD^2 + GE^2 + 2\ DG\ GE$
$CE^2 = (DG - GE)^2 = DG^2 + GE^2 - 2\ DG.GE$
$AE^2 = (BF - EF)^2 = BF^2 + EF^2 - 2\ BF.FE$
$BE^2 = EF^2 + BF^2 + 2\ BF\ FE.$
$ED^2 + CE^2 + AE^2 + BE^2 = 2\ (GD^2 + EF^2) + 2\ (GE^2 + BF^2)$
$= 2\ OD^2 + 2\ OB^2$
$= 4r^2 = (2r)^2.$

1887-88

4. $AD^2 = AC^2 + CD^2 + 2\ CD\ CE$
$= AC^2 + CD.(CD + 2\ CE)$
$= AC^2 + BD\ DC$
$- 2\ AC^2$

7. $OP^2 = OQ\ OR = OQ.(2\ OQ) = 2\ OQ^2 = 2\ QR^2.$
But $OP_2 = 4\ r^2$
$\therefore$ $QR^2 = 2\ r^2 = CQ^2 + CR^2$
$\therefore$ $\angle C$ is a rt $\angle$ [I 48

9 (AB+AC) − AC
=DB+BE=2 BE=2 OD=diameter,
[ODBE is a square.]

1890-91

1 *(a)* AB+AC>BD+DC,
AC+CB>AD+DB,
CB+BA>CD+DA,
2 perimeter >2 (AD+DB+DC)
∴ perimeter >(AD+DB+DC)

2 Draw EK is ||l to AB, and GH is ||l to HD, and FL is ||l to AD

Now the fig GC=2 Δ FEG , the fig GB=2 Δ EAG ,
∴ the fig HC=2 Δ AEF

Now fig GD=fig GL
∴ fig LK=fig AF, or rect BE DF=fig AF,
2 Δ AEF + rect BE DF=rect ABCD.

6 *(b)* On AB, describe the segment ADB containing an ∠ = ∠ E From C draw CD ⊥ to AB, meeting ADB in D Join BD, DB Then ADB is the Δ reqd

Join BD

∠ BDG= ∠BCA= ∠ BLG [Since the fig. CH is cyclic
∠s at G are rt ∠s, and BG is common,
LG=DG

Similarly, LF LE are respectively, bisected by AB, AC

1891-92

In the Δs AOB, DOC,
AB=DC, AO=CO,
BHO= ∠OCD , ∴ ∠ AOB= ∠COD
∴ BO, OD are in the same st line [I 15
Δ ABC=½ CO AB,
Δ ADB=½ ODOB,
∴ the quadtl ACBD=½ AB CD=½ AB²

5 $AB^2+BH^2=AH^2+2$ AB BH
$=AH^2+2\ AH^2=3\ AH^2$

7 Chords AC, AD are equal,
∴ arcs AC, AD are equal,
∴ ∠ ACD= ∠ AEC,
∴ ∠ DFE= ∠ACE, [adding ∠ ECF

8. See Bombay Paps 1887-88, q (9).

1892-93

3 EF is parallel to AN, and also to AM

∴ NA and AM must be the same st line

5. $BC^2 = CD^2 + CD^2$

$= AB^2 - AD^2 + CD^2$

$= (AC^2 - AD^2) + CD^2$

$= CD^2 + 2\,AD.\,DC + CD^2$

$= 2\,CD\,(CD + AD)$

$= 2\,AC\;CD$

8 Bisect ∠ A by AF, join CF

∠ FCB = ∠ FAB = ½ ∠ DAB = ½ ∠ BCE

∴ FC bisect ∠ BCE

9. Let OP, OR be the roads; P, and Q their houses Now the locus of points equidistant from P and Q, is the st line DE bisecting OQ at rt ∠s And the locus of points equidistant from PO and RO, is the st line OC, the bisector of the ∠ POR

∴ E, the intersection of the loci, is the point reqd.

APPENDIX.

A CLASSIFIED INDEX

TO

THE FIRST FOUR BOOKS

OF THE

ELEMENTS OF EUCLID.

THEOREMS.

A. Of the Angles formed by the Meeting and Intersection of Straight Lines

	HYPOTHESIS	CONSEQUENCES.
I. 13 ...	If a straight line standing upon another, forms angles with it	They are either two right angles, or are together equal to two right angles.
I. 14 ...	If two straight lines meet another straight line at the same point and on opposite sides, and make the adjacent angles with it together equal to two right angles	Those two straight lines will form one continued straight line
I. 15. ...	If two straight lines intersect	The vertical angles are equal

B. Of Parallel Straight Lines

	HYPOTHESIS	CONSEQUENCES
	If a straight line intersect two other straight lines, both in the same plane.	The two straight lines shall be parallel
I 27 ...	And form *alternate* angles equal to each other.	
I. 28. A ...	Or form an exterior angle equal to the interior and opposite angle upon the same side of the line	
I. 28 B ...	Or form interior angles at the same side equal to two right angles	

		HYPOTHESIS	CONSEQUENCES
I	29	If a straight line intersect two parallel straight lines	It forms the alternate angles equal to one another, And the exterior angle equal to the interior and opposite angle upon the same side, And also the two interior angles on the same side, together equal to two right angles
I	30	If two straight lines be parallel to the same straight line	They are parallel to each other
I	33	If two straight lines join the adjacent extremeties of two equal and parallel straight lines	They are themselves equal and parallel

C *Comparison of Triangles as to Equality*

		HYPOTHESIS	CONSEQUENCES
I	7	If two triangles be upon the same base, and on the same side of it	They cannot have their sides which are terminated in one extremity of that base equal to one another, and also those which are terminated in the other extremity
I	26	If two triangles have two angles in the one respectively equal to two angles in the other, And a side of the one equal to a side of the other, either the sides *adjacent* to, or the sides *opposite* to, those equal angles	The remaining angles and sides shall be respectively equal to one another And the triangles themselves shall be equal to one another

	HYPOTHESIS	CONSEQUENCES
I. 8, and cor	If two triangles have two sides of the one respectively equal to two sides of the other, And have also their bases equal	The angles formed by the equal sides are equal. And the angles opposite the equal sides are equal And the triangles themselves are equal
I 25	But if the *third* side of the one be greater than the *third* side of the other	The angle opposite to the greater side is greater than the angle which is opposite to the less
I 4	If two triangles have two sides of the one respectively equal to two sides of the other, And the angles formed by those sides also equal to one another	The triangles themselves will be *identically* equal
I 24	But if the angle formed by two sides of one be greater than the angle formed by the two sides equal to them of the other	The side opposite to that greater angle is greater than the side which is opposite to the less.
I 37	If triangles are between the same parallels And upon the *same* base,	They are equal to one another in area
I 38	Or upon *equal* bases	
	If triangles are equal in area	
I 39	And upon the *same* base, and *on the same side of it*	They are between the same parallels
I 40	Or upon *equal* bases in the same straight line, and on the same side of it	

D On the Relations between the sides and Angles of Triangles.

	HYPOTHESIS.	CONSEQUENCES
I 20 .	If two straight lines are the sides of a triangle.	They are together greater than the third side
I 17.	If any two angles are those of a triangle	They are together less than two right angles
I 32 B .	If any three angles are the interior angles of a triangle	They are together equal to two right angles
I. 6 ..	If two angles of a triangle are equal	The sides opposite to those angles are also equal
I 19	If in any triangle one angle is greater than another.	The side which is opposite to the greater angle is greater than the side which is opposite to the less
I 5 A	If a triangle be *isosceles*	The angles *at the base* are equal to one another
I 18.	If one side of any triangle be greater than another	The angle opposite to the greater side is greater than the angle which is opposite to the less
I 16	If one side of a triangle be produced.	The exterior angle is greater than either of the internal opposite angles
I 32 A .	If one side of a triangle be produced	The exterior angle is equal to the sum of the two interior and opposite angles
I 47 ..	If a triangle be right-angled.	The square which is constructed upon the side subtending the right angle is equal in area to the sum of the squares constructed upon the sides which form the right angle

	HYPOTHESIS	CONSEQUENCES
I 48 ..	If the square constructed upon one side of a triangle be equal in area to the sum of the squares constructed upon the other two sides	The angle opposite to that side is a right angle

E On the Relations of Lines drawn in Triangles

	HYPOTHESIS.	CONSEQUENCES
II. 12. .	If a perpendicular be drawn from any of the acute angles of an obtuse-angled triangle to the opposite side produced	The square on the side subtending the obtuse angle is greater than the sum of the squares on the two sides which contain the *obtuse* angle, by *twice* the rectangle under the side, which is produced, and the external segment between the obtuse angle and the perpendicular
II. 13 .	If in any triangle a perpendicular be drawn to one of the sides which contains an acute angle, from the opposite angle	The square on the side subtending that *acute* angle is less than the sum of the squares on the sides which contain that angle, by twice the rectangle under the side to which the perpendicular is drawn, and the segment between the perpendicular and the acute angle
I 21. .	If from a point within a triangle, two straight lines be drawn to the extremities of any side	They are together less than the sum of the two other sides of the triangle And they form a greater angle.

F Comparison of Parallelograms with Triangles.

	HYPOTHESIS	CONSEQUENCES
I 41. ..	If a parallelogram and a triangle be between the same parallels And upon the same base.	The parallelogram is double the triangle

G Comparison of Parallelograms as to Equality

	HYPOTHESIS	CONSEQUENCES
I 35	If parallelograms are between the *same parallels*, And *upon the same base*,	They are equal in area.
I 36. ..	Or upon *equal* bases	

H On the Relations between the Sides, Angles, and Surfaces of Parallelograms

	HYPOTHESIS	CONSEQUENCES
I 34 .	If a figure be a parallelogram	The opposite *sides* are equal to one another The opposite *angles* are equal to one another And the parallelogram is bisected by its diagonal.
I 43. ...	If about the diagonal of a parallelogram two other parallelograms are formed	Their *complements* are equal in area.

I. *Comparison of Rectangles contained by Straight Lines and their Segments.*

	HYPOTHESIS.	CONSEQUENCES.
II. 2 ...	If a straight line be divided into any *two* parts	The rectangles under the whole line, and each of the parts, are together equal in area to the square on the whole line.
II 3. ...	If a straight line be divided into any *two* parts	The rectangle under the whole line, and *one* of those parts, is equal in area to the square on *that part* together with the rectangle under the two parts
II. 4. ..	If a straight line be divided into any *two* parts	The square on the whole line is equal in area to the sum of the squares on the parts, together with *twice* the rectangle under the parts
II 7. ...	If a straight line be divided into any *two* parts	The sum of the squares on the whole line and the square on *either segment* is equal in area to twice the rectangle under the whole line and that segment, together with the square on the *other segment*
II 8. ...	If a straight line be divided into any *two* parts	The square on the sum of the whole line and either segment is equal in area to *four* times the rectangle under the whole line and that segment, together with the square on the other segment

	HYPOTHESIS.	CONSEQUENCES.
II 5 ..	If a straight line be *bisected*, and also *cut into two unequal parts*	The rectangle under the unequal parts together with the square on the *line between the points of section*, is equal in area to the square on *half* the line
II 9 .	If a straight line be *bisected*, and also *cut into two unequal parts.*	The sum of the squares on the unequal parts is equal in area to twice the sum of the squares on *half* the line and the square on the line between the points of section.
II 6 ..	If a straight line be *bisected*, and also *produced to any point*	The rectangle under the whole line thus produced and the part produced, together with the square on half the *line bisected*, is equal in area to the square on the straight line which is made up of the half and part the produced
II 10 ..	If a straight line be *bisected*, and also *produced to any point.*	The square on the whole line thus produced, together with the square on the part produced is equal in area to twice the square on *half* the line bisected, together with twice the square on the straight line, which is made up of the half and the part produced

	HYPOTHESIS.	CONSEQUENCES.
II 1. ...	If there be two straight lines, one of which is divided into any number of parts.	The rectangle under the two lines is equal in area to the sum of the rectangles under the *undivided* line and the several parts of the *divided* line.

K *Of Polygons.*

	HYPOTHESIS	CONSEQUENCES
I. 32, Cor. 1.	If a figure be rectilineal	The sum of all the *interior* angles, together with *four* right angles, is equal to twice as many right angles as the figure has sides
I. 32, Cor. 2	If a figure be rectilineal.	All its *exterior* angles are together equal to *four* right angles.

L *Relative to circles generally.*

	HYPOTHESIS	CONSEQUENCES.
III. 2. ...	If any two points be taken in the circumference of a circle	The straight line which joins them falls within the circle

M *On the Relations of Straight Lines drawn from any Point to the Circumference of a Circle.*

	HYPOTHESIS.	CONSEQUENCES
II 7. ...	If from any point *within* a circle, which is not the centre, straight lines be drawn to the circumference.	The *greatest* is that which passes through the centre The remaining part of the diameter is the *least*. That line which is nearer to the line passing through the centre is *greater than one more remote*. And more than two straight lines cannot be drawn which shall be equal.

	HYPOTHESIS.	CONSEQUENCES
III 9 ..	If a point be taken within a circle, from which more than two equal straight lines can be drawn to the circumference	That point is the centre of the circle.
III 8 .	If from any point *without* a circle straight lines be drawn to the circumference.	Of those which fall on the *concave* circumference, the *greatest* is that which passes through the centre. Of the *rest*, that which is *nearer* to the line passing through the centre, is *greater* than the more remote. But of those, which fall on the *convex* circumference, the *least* is that which, if produced, would pass through the centre. Of the *rest*, that which is nearer to the least, is less than the more remote. And more than two straight lines cannot be drawn, either to the *concave* or *convex* circumference, which shall be equal
III 36. ..	If from a point *without* a circle, two straight lines be drawn, one of which cuts the circle, and the other *touches* it	The rectangle under the whole line which cuts the circle and the *segment without* the circle is equal in area to the square on the line which *touches* it

	HYPOTHESIS	CONSEQUENCES
III 37 .	If from a point *without* a circle, two straight lines be drawn, one cutting the circle and the other meeting it, and if the rectangle under the whole line which cuts the circle and the part of it without the circle, be equal in area to the square on the line which meets it	That st. line is a *tangent* to the circle
III 4 .	If in a circle, two straight lines cut one another, which do not both pass through the centre	They do not bisect each other
III 35 .	If two straight lines cut one another *within* a circle	The rectangle under the segments of one of them is equal in area to the rectangle under the segments of the other.
III 3. .	If a straight line be drawn through the centre of a circle, bisect a straight line which does not pass through the centre	It is *perpendicular* to it
	And if it is *perpendicular* to it	It *bisect* it
III 18 .	If a straight line *touches* the circumference of a circle	The straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.
III 19 .	If a straight line *touches* the circumference of a circle, and a straight line be drawn perpendicular to it from the point of contact.	The centre of the circle shall be in *that* line

	HYPOTHESIS	CONSEQUENCES.
III 16. ...	If a straight line be drawn from the extremity of the diameter of a circle, perpendicular to the same.	It will be a *tangent* to the circle.
	And if any straight line be drawn from a point between that perpendicular and the circle, to the point of contact	It will *cut* the circumference.

O. *On the Mutual Contact of two Circles*

	HYPOTHESIS	CONSEQUENCES
III. 10 ...	If two lines be the circumferences of two circles	They cannot cut one another in more than *two* points
III 5 ...	If the circumferences of two circles *cut* one another	They have not the *same centre*
III 6 .	If the circumference of one circle *touch* the circumference of another circle *internally* in any point	They have not the *same centre*
III 11 ...	If the circumference of one circle touch the circumference of another circle *internally* in any point	The straight line joining their centers, being produced, shall pass through *that* point
III 12 .	If the circumference of two circles touch each other *externally* in any point.	The straight line joining their centers shall pass through *that* point.
III 13 .	If the circumference of one circle touch the circumference of another circle, either *internally* or *externally*	There can only be one point of contact

P. *On the Angles in a Circle*

	HYPOTHESIS.	CONSEQUENCES
III 26 ...	If equal angles are in *equal* circles, or in the *same* circle	They shall stand upon equal arcs whether they be at the centre or the circumference
III 27 ...	If angles stand upon equal parts of the circumferences of *equal* circles, or of the *same* circle	They are equal to one another, whether they be at the centre or the circumference
III. 21 ...	If angles are in the same segment of a circle	They are equal to one another.
III 31. ..	If, in a circle, an angle be in a *semicircle*	It is a *right* angle
	But if the angle be in a segment *greater* than a semicircle	It is *less* than a right angle.
	And if the angle be in a segment *less* than a semicircle	It is *greater* than a right angle
III. 20 .	If an angle at the centre of a circle have the same part of the circumference for its base as an angle at the circumference	The former angle is *double* the latter
III 32. ..	If a straight line *touches* a circle, and from the point of contact, a straight line be drawn cutting the circle	The angles formed by this line and the line touching the circle are equal to the angles in the *alternate segments* of the circle

Q *On Segments and their Chords*

	HYPOTHESIS	CONSEQUENCES
III 23	If two segments of circles are upon the same straight line, and upon the same side of it	They cannot be similar without coinciding with one another
III 24	If two segments of circles are *similar*, and upon equal straight lines	They are equal to one another, and have equal *arcs*
III 14	If two *chords* in a circle are equal And chords which are equally distant from the centre	They are equally distant from the centre They are equal to one another
III 15	If chords be drawn in a circle, of which one passes through its centre	That chord is the greatest And of all others, that which is nearer to the centre is greater than the more remote And the greater is nearer to the centre than the less
III 28	If, in equal circles, or the same circle, straight lines are equal.	They cut off equal parts of the circumferences, the greater, equal to the greater, and the less to the less
III 29	If in equal circles, or the same circle, equal parts of the circumference are taken	They are subtended by equal chords

R *On Figures contained in Circles*

	HYPOTHESIS	CONSEQUENCES
III 22	If a four-sided figure is contained within a circle.	Its opposite angles are together equal to two right angles

PROBLEMS

A Relating to Straight Lines.

I 2.	...	From a given *point*, to draw a straight line equal to a given *finite straight line*
I. 31	...	Through a given *point*, to draw a straight line parallel to a given *straight line*
I 3.	..	From the greater of *two* given *straight lines*, to cut off a part equal to the less
I 10	...	To bisect a given *finite straight line*
II. 11	..	To divide a given *finite straight line* into two parts, so that the rectangle under the whole line and one segment shall be equal in area to the square on the other segment

B Relating to rectilineal Angles

I 23.		At a given *point* in a given *straight line*, to make a rectilineal angle equal to a given *rectilineal angle*
I 9.	..	To bisect a given *rectilineal angle*
I 11.	..	From a given *point* in a given *straight line*, to draw a perpendicular to that line.
I 12	..	To draw a straight line perpendicular to a given *straight line of an unlimited length*, from a given *point without it*

C Relating to Triangles

I 22.	...	Given *three finite straight lines, of which any two together are greater than the third*, to construct a triangle whose sides shall be respectively equal to the given lines
I 1		To construct an equilateral triangle upon a given *finite straight line*

D Relating to Parallelograms

I 42	...	To construct a parallelogram equal in area to a given *triangle*, and having an angle equal to a given *rectilineal angle*
I 44.	.	Upon a given *finite straight line* to construct a parallelogram equal in area to a given *triangle*, and having an angle equal to a given *rectilineal angle*
I 45	...	To construct a parallelogram equal in area to a given *rectilineal figure*, and having an angle equal to a given *rectilineal angle*.
I 46	...	Upon a given *finite straight line*, to construct a square
II. 14.	.	To construct a square, equal in area to a given *rectilineal figure*.

E. *Relating to Circles*

III.	1.	...	To find the centre of a given *circle*
III.	17	...	From a given *point, either without a given circle or in its circumference*, to draw a straight line *touching* the circumference.
III.	30.	..	To bisect a given *arc*
III	25	.	*A segment of a circle* being given, to describe the circle of which it is a segment
III.	33	..	On a given *finite straight line*, to describe a segment of a circle, which shall contain an angle equal to a given *rectilineal angle*
III	34	...	To cut off from a given *circle* a segment which shall contain an angle equal to a given *rectilineal angle*.

BOOK IV

A *Relating to Triangles*

IV.	10	.	To construct an isosceles triangle, in which each of the angles at the base, shall be double of the vertical angle

B *Relating to Inscribed Figures.*

IV	1	...	In a given *circle*, to place a straight line equal to a given *straight line*, which is not *greater* than the *diameter* of the circle.
IV	4	.	To inscribe a circle, in a given *triangle*
IV	5	..	To circumscribe a circle about a given *triangle*
IV.	2	...	In a given *circle* to inscribe a triangle equiangular to a given *triangle*
IV	3	..	About a given *circle*, to circumscribe a triangle equiangular to a given *triangle*
IV	8	.	To inscribe a circle in a given *square*
IV.	6	..	To inscribe a square in a given *circle*
IV	9	.	To circumscribe a circle about a given *square*.
IV	7	...	To circumscribe a square about a given *circle*
IV.	13.	...	To inscribe a circle in a given *equilateral and equiangular Pentagon*
IV.	14	...	To circumscribe a circle about a given *equilateral and equiangular Pentagon*
IV.	12	.	To circumscribe an equilateral and equiangular *Pentagon* about a given *circle*
IV.	11	...	To inscribe an equilateral and equiangular *Pentagon* in a given *circle*
IV.	15.	.	To inscribe an equilateral and equiangular *Hexagon* in a given *circle*
IV.	16	.	To inscribe an equilateral and equiangular *Quindecagon* in a given *circle*

CALCUTTA UNIVERSITY PAPERS

ENTRANCE EXAMINATION.

Geometry.

1858

1 If one of the acute ∠s of a rt ∠d Δ be=2ce the other, the hypotenuse is 2ce the shorter side.

2 If any pt be taken within an equilateral Δ, the sum of the ⊥rs drawn from it to the sides, is=to the ⊥r from the vertex to the base

3 II 10

1859 (I)

1. Shew that the diagonals of a rhombus bisect one another and cut at rt. ∠s

2 In any Δ ABC, if the ∠s at A, B be bisected by st. lines which meet at D, shew the line joining D and C will bisect ∠ ACB (IV 4 E 2, Text p. 203.)

3 The sqs on the diagonals of a parallelogram=the sum of the sqs on the 4 sides

4 I. 13; 5 I 23, 6 I 27, 7 I 48, 8 II 5, 9 III 4.

1859 (II)

1. From the same point, there cannot be drawn more than two equal straight lines to meet a given straight line.

2. Prove that the 4 Δs into which a parallelogram is divided by its diagonals, are equal

3. If two chords of a ○ intersect at rt ∠s, the portions of the ⊙ce taken *alternately*, are together=to the semi ○ce

4. If two ○s cut one another, find a pt. from which the st lines drawn to touch the two ⊙s shall be equal.

5 I 16, 6 I 34, 7 II 12, 8 III 20, 9. III 32.

1860.

1 If in the fig. of I. 5, H be the pt. of intersection of BG, CF, prove that AH will bisect ∠BAC

2 From a given pt. draw a st line making equal ∠s with two given st lines

3 If on the radius AO of a ○, whose centre is O, a semi-○ be described, and from any pt. M in AO, a st line be drawn at rt ∠s to it, cutting the semi ○ at P and the larger ○ at Q, and if AP, AQ be joined, shew that $AQ^2 = 2\ AP^2$

4 Any ∠ of a Δ inscribed in a ○ is > or < a rt ∠, by the ∠ contained by the side subtending the ∠, and a diameter from either extremity of that side (Use III 31 and III 21)

5 I. 31, 6 II 14, 7 III 21, 8 III. 34

1861

1. Thro' a given pt draw a st line which shall make equal ∠s with two st lines given in position

2 If the st line bisecting the vertical ∠ of a Δ, also bisects the base, the Δ is isosceles (Use I 8)

3 The sum of the sqs on the sides of a parallelogram, is equal to the sum of the sqs on the diagonals

4 Given the ∠ at the base of an isos Δ, and the ⊥r from it on the opposite side, construct the Δ

5 I 47, 6 II. 5, 7 II 10, 8 III 27, 9 III 31

1862

1 Construct an isos Δ, whose external vertical ∠ is 67½ degrees

2 In the side BC of a rt ∠d Δ ABC, rt. ∠d at C, find a point D, such that the ⊥r DF drawn D to the hypotenuse shall be = AF

3 The area of a rhombus is = to half the rect contained by the diagonals

4 Given a chord AB of a ○, and a point C in it, find in the ○ce, a point D, such that the line DC shall bisect the vertical ∠ of the Δ ABD

5 I 6, 6 Cor 1 of 32, 7 II 13, 8 III 32, 9 IV 16

1863

1. Given two equal and parallel st lines AB, DC prove that AC, BD bisect each other Under what circumstances will AC, BD be equal ?

2. Three st lines meet at a pt. Draw another line cutting them, so that the segment of it intercepted between the 1st and the 2nd, shall be = to that intercepted between the 2nd and 3rd

3 Describe a square that shall be = to a given Δ. (II. 14)

4 What is the *locus* of the middle pts. of equal st lines in a ◯?

5. A tangent is drawn parallel to a chord, shew that the intercepted arc is bisected at the pt of contact

6 I 4, 7. II 14, III 22, 9 IV 11

1864

1 Show that every four sided fig whose opposite sides are equal, is a parallelogram

2 In a rt ∠ d Δ, the line joining the rt. ∠ to the pt of bisection of the hypotenuse, is = to half the hypotenuse

3 I 20, 4 I 32, with 2 Cors 5 II 9, 6 III 32, 7 IV 12

1865

1 Given one of the sides of a rt ∠ d Δ containing the rt ∠, and the *sum* of the other two sides, construct the Δ

2. Given one of the sides of a rt. ∠ d Δ containing the rt ∠, and the *difference* of the other two sides, construct the Δ

3. The st line drawn from the rt ∠ of a rt ∠ d Δ, to the middle pt of the opp side, is = to half that side

4 Divide a given st line into two parts, so that the rectangle, contained by them shall be = a given square.

5 Produce a given st. line, so that the rect contained by the whole line thus produced and the part of it produced, shall be = a given square

6 If a rectilineal fig of an *even* number of sides, be *inscribed* in a ◯, the 1st, 3rd, 5th, &c, are together = to the 2nd, 4th, 6th, &c, ∠s taken together, any ∠ being assumed as the 1st (See Text, p 116)

7 If a ⊙ be *inscribed* in any Δ, the points of contact shall divide the sides into segments such that any one side together with the *remote segment* of either of the other two, shall be = ½ the sum of the sides.

8 I 24, 9 II 5, 10. III 21

1866 (I.)

1 AB is parallel to CD and unequal to it, and they are joined towards the same parts by AC and BD If AC=BD, shew that AD=BC

2. Describe a ○ wh shall touch a given st line, and pass thro' another given point

3 I 20, 4 I 33, 5 III 2; 6 III 31; 7. III 35, 8. IV 4, 8. IV 13.

1866 (II).

1 Produce a given st line to a point, such that the rect. contained by the whole line thus produced, and the part produced, shall be=to the square on a given line

2 ABC is an isosceles Δ of which B is the vertex, BA, BC are bisected at D and E respectively; AE, CD, intersect at F Shew that Δ BDE=3 Δ DEF

3 Construct a rectangle that shall be=to a given square, the difference of two adjacent sides being given

4. If a tangent of a (be parallel to a chord, prove that the intercepted arc is bisected at the pt of contact

5 Describe a ○ that shall touch a given line and a given ○ (Ex 9 Text, p 238)

6 I 7; 7 I 43, 8 II 7, 9 II 11, 10 III 13; 11 IV. 10

1867

1. Construct an isos Δ having each of the sides 2ce of the base

2 The st line which bisects the vertical ∠ of an isos Δ, bisects the base ⊥ rly

3 Describe a rhombus = to a given square.

4 I. 6, 5 I 35, 6 II 4, 7 II. 14, 8. III 20, 9 III 35, 10 IV 8 and 10

1868

1. Prove that the 3 interior ∠s of every Δ are=2 rt ∠s, *without producing a side* (See Notes on I. 32.)

2 Shew from I 47, how to find a square which shall be equal to the *difference* of two given squares

3. Prove by II. 12 and II 13, that if any side of a Δ be bisected the sqs. on the other two sides are together equal to 2 sq on

the line drawn from the pt of bisection of the third side to the opp ∠ + 2 sq. on half the line bisected (Ex 24 Text p 147.)

4. Shew, by assuming that the ∠ in a semi ⊙ to be a rt. ∠, how III 17 may be more *simply* effected (Text, p 202.)

5 The three pts of contact of a ⊙ *inscribed* in a Δ are joined, shew that the resulting Δ is acute ∠d

6 Construct a rt ∠d Δ, having given the hypotenuse and the sum of the sides (Text, p 107)

7. Two ⊙s have the same centre, shew that all chords of the *outer* ⊙ which touch the *inner* ⊙ are equal

8 I 22, 9 II. 8 · 10 III 17, 11 III 33; 12. IV. 4

1869

1 Having given the base of a Δ, the difference of the sides and the difference of the ∠s at the base, describe the Δ (Text, p 108)

2. If two ⊙s intersect one another, their common chord when produced, bisects their common tangent.

3 Inscribe a ⊙ in a rhombus

4 I. 44; 5 I 48, 6 II 12, 7. III 16, 8 IV 4.

1870

1 Given that Δs of equal area are between the same parallels, prove that their bases are equal.

2 Two lines OA, OB being given, intersecting at O and a point C in OA, describe a ⊙ touching OA at C, and also touching OB.

3 The line drawn from the rt. ∠ in a rt ∠d Δ, to the bisection of the hypotenuse = ½ the hypotenuse.

4 I 32 with Cor 2, 5 II. 11, 6 III. 31

1871.

1 In a polygon of n sides, the sum of all the interior ∠s = $(2n-4)$ rt. ∠s.

2. Find a line whose square shall be = to the sum of the sqs on 3 given lines

3 ABC is a rt ∠d Δ, AD the ⊥r from A on the hypotenuse BC, is produced in the direction DA till it meets a side produced of the square on AC in O. Prove, that it will meet a side pro-

duced of the sq on AB in the same point O, that AO shall be = BC, and that if O be joined with B, and A with E, the extermity of the side BE of the sq on BC, the fig OAEB shall be a parallelogram, and = the sq on AB

4 Given the base, vertical ∠, and the ⊥r, let fall from the vertex on the base, construct the Δ, and shew that, in general, there can be two Δs satisfying the given conditions

5 The ⊥rs erected at the middle pts of the sides of a Δ, meet at a point (Text, p 106 and p 224)

6 I 32, 7 I 48, 8. II 1, 9 III 32; 10 III 36, 11 IV 5

1872

1 Divide a line so that the rect contained by the parts shall be the *greatest* possible

2 In a Δ APB, AP^2 is < BP^2 by a *constant* quantity Prove that P must be on a certain st line

3 If the parts of two chords at rt ∠s to one another be given, explain how the length of the radius of the ○ may be calculated

4 Compare the area of a regular *hexagon* inscribed in a ○ with that of an equilateral Δ inscribed in the same ○

5 Express each ∠ of a regular *pentagon* and of a regular *decagon* in terms of a rt ∠

6 I 48, 7 II 9, 8 II 13, 9 III 4, 10 III 36, 11 IV 15; 12. IV 16

1873

1 If the middle pts of the sides of a Δ be joined the Δ so formed shall be equiangular to the given Δ, and = *one-fourth* of it

2 The external ∠s DBC, ECB of a Δ ABC are bisected by BE, CF, and FG, FH, are drawn ⊥rs to AD, AE, prove that FG = FH and AG = AH (Text, p 255)

3 AB is a chord of a ○, C a pt in the ○ce of the smaller segment find a point D in the ○ce of the larger, show that AB shall bisect ∠DBC

4 I 4, 5 I 45, 6 II 13, 7 III. 11, 8 III. 33

1874

1 Deduce from II 4, that the square on a given line = 4 times the square on its half

2. Prove that if an ∠ of a Δ be *two-third* of a rt ∠, the sq on the side opp to it = the sum of the sqs. on the side containing it, *diminished* the rect contained by them

3 State, *without proving*, the conditions which must be fulfilled in order that a ⊙ may be described so as to pass (1) thro' *two* given points, (2) thro' *three* given points, (3) thro' *four* given pts

4 A ⊙ is described so as to touch the side BC of a Δ ABC at D and AB, AC produced, at E and F, show that Δ EDF is obtuse ∠d. (See q 5 of 1868)

5 QA and QB are two straight lines in a ○ at rt ∠s to one another, QD is a diameter, P any point in the ○ of the smaller segment cut off by QA, show that Δ APQ + Δ BQP = Δ QPD

6 I 5, 7 I 24; 8 II. 4, 9. III 13, 10 III 31; 11 IV 13, 12 IV 15

1875.

1 If two lines are equal and parallel, shew that if the extremities be joined "*not towards the same parts*," two equal Δs will be formed

2. Shew how the enunciation of II 9, may be made to include II. 10 (See Notes on II. 9 and II. 10)

3 If from any pt without a ⊙, st lines be drawn touching it, the ∠ contained by the tangents is = 2 ∠ contained by the st line joining the points of contact and the diameter through either of them

4. I 33, 5 II 9 and 10; 6. III. 20 · 7 IV 15.

1876

1. AB, CD are two st lines intersecting at O, CA, DB are ⊥rs to AB, OB is = 2 OA Prove that OD is = 2 OC

2 C is the centre of a given ⊙, A any other pt. within it, AB is drawn at rt ∠s to AC and meets the ○ce at B Prove that the ○ about Δ ABC touches the given ○, and that ∠ ABC is the *greatest* ∠ subtended by AC at any point in the ⊙ce of the given ○

3 I. 42; 4 II. 12; 5 III. 17, 6. III 35, 7. IV 9.

1877

1 Draw a common tangent to two given circles. (See Text, p 218.)

2 State what regular polygon has each of its ∠s = *nine-tenths* of 2 rt ∠s

3 I 24, and Cors I 32, 4 II. 7, 5 III 16, 6 IV 15.

1878

1 Describe a ⊙ touching one side of a Δ and the other two produced (Text, p 255)

2 ABC is a Δ with a rt ∠ at A, AD is ⊥r to BC, to what rectangle is the square on AD equal?

3. III 11, 4 IV 10

1879

1 In two ⊙s which touch each other *externally*, two parallel diameters are drawn Shew that one extremity of each diameter and the point of contact lie in the same st line

2 A ⊙ is described to touch BC, a side of the Δ ABC an D, and the other two sides produced at E and F respectively Prove that AF = ½ sum of the sides of the Δ ABC

3 Two fixed points A, B lie on the same side of a fixed line CD of *unlimited length* P is any point in CD Prove that AP+BP is *least* when the ∠s which AP, BP make with CD are equal (Text, p 243)

4 I 24, 5 II 11, 6 IV 4

1880

1 The side BC of a Δ ABC is produced to D, shew that ∠ ACD is > ∠ ABC without shewing that it is > ∠BAC

2. AOC, BQD are two Δs, having ∠ AOC = ∠BQD, and ∠ACO = ∠DBQ, shew that the rect AO QB = rect. CO QD (See Notes on III 35)

3 Shew that the square on the side of an equilateral Δ *described* about a ⊙, is 4 times the square on the side of an equilateral Δ *inscribed* in the same ⊙

4 I 6, 5 I 48; 6 II 11, 7 III 11, 8 III 35, 9 IV 5

1881.

1 Shew how to make a Δ = a given quadrilateral which shall have its base on one side of the quadrilateral produced If necessary, and its vertex at one of the opposite ∠s

2 BC is a given arc of a ⊙, whose centre is O; A is any pt. in BC; AD, AE are drawn ⊥rs to OB, OC; prove that the line DE is of *constant* length

3 I. 17; 4 I 22; 5 I. 37; 6 II. 9, 7. III 22, 8 IV 11.

1882

1 OC is a st line which bisects ∠ AOB, OD is any other st. line *without* the ∠ AOB, shew that the ∠s DOA, DOB are = 2 ∠ DOC

2 ABC is a Δ, st lines AD, CE, bisect the ∠s at A and C, BE is drawn from B = BC, and BD = BA, shew that EBD is a straight line

3 If A, B be fixed pts and O any other pt, the sum of the sqs on AO and BO is *least*, when O is the *middle point* of AB

4. If two st lines, AB, CD in a ⊙ intersect at E, the ∠ subtended by AC and BD at the centre are together = 2 ∠ AEC.

5. AO, BO are radii of a ⊙ at rt ∠s to each other, ACD is a st. line meeting OB in C, and the ⊙ in D Then rect. AC. AD = 2ce the square on OB

6 I 9 and 27; 7. II. 9 8 III. 20, 9 III 36; 10 IV 2

1883.

1. ABC is a Δ The line bisecting ∠ B meets the line bisecting ∠ C at the point G, and the line bisecting the *external* ∠ at A at the point D. Prove that ∠ADG = ∠ACG.

2 Through one extremity of the common chord of two intersecting ⊙s, two st lines are drawn terminated by these ⊙s Prove that the lines joining the other extremity of the common chord and the two terminal pts of the two st lines on each ⊙ together with the lines joining these terminus pts, from two equiangular Δs.

3. I. 24 4 I 42, 5 II 13; 6 III 35, 7. IV. 2.

No Examination in 1884

1885

1. Divide a given st line into two parts, such that the difference of the sqs. on the two parts may be = sq on a given line.

2 AB is a diameter of A and AC, the ⊙ tangent at A = in length to A B, CB is joined cutting the ⊙ at D, prove that CB is bisected at D, and AD = ½ CB

3 Describe a ○ touching three given st lines, no two of which are parallel Shew that 4 such ○s can be described (Text, p 255)

4 ABC is an acute ∠d Δ; ⊥s AD, BE, CF, are drawn from A, B, C, upon the opposite sides respectively, intersecting at O, prove that O is the centre of the ○ inscribed in the Δ DEF, and A, B, C are the centres of the ○s *escribed* to the same Δ

5 I. 14, 6 I 47, 7 II 12, 8 III 31, 9 IV 15

1886.

1 If a quadrilateral has two opposite sides equal and parallel, it is a parallelogram.

2 The sq on the difference of two st lines + twice the rect contained by the two lines = sum of the sqs on these lines

3 (*1*) If two ○s touch *internally*, the centre of the inner ○ lies on that radius of the outer ○ which passes through the point of contact (III. 11)

(*2*) Also shew that any chord of the outer ○ drawn from the point of contact is bisected by the inner ○, if the ○ passes through the centre of the outer ○

4 With the aid of an isosceles Δ such that each of the ∠s at its base is 7 times the ∠ at the vertex, inscribe a regular *quindecagon* in a given ○ Give geometrical proof

5 The sum of the *medians* of a Δ is less than the sum of the sides

6 Two equal ○s intersect in A, B. Let CD and EF be chords of the ○s each = chord AB, and so placed on opposite sides of AB, that all the three chords meet in H Then AH bisects ∠CHE

7 I 33, 8 II 7, 9 III 11

1887.

1 ABCD is a quadrilateral, of which the sides AB, DC are parallel, E, F are the middle points of the sides BC, AD respectively, prove that EF is parallel to AB or CD and = ½ their sum

2 If two ○s cut each other, their common chord produced bisects their common tangents

3 In what case does the centre of the inscribed ○s, coincide with that of the circumscribed ○, and why?

4 AB, AC are tangents to a given ⊙, BC the chord of contact. From the middle point D of BC, the st line EDF is drawn at rt ∠s to BC, cutting the Oce of the given ⊙ at E, F. Prove that E, F are the centres of two ⊙s one of which touches the three sides, and the other touches one side and two sides *produced*, of the Δ ABC.

5 I 32 and 48, 6 II. 9; 7. III 37, 8 IV 4

1888.

1 Shew that the area of a Δ = ½ the rectangle contained by its base and the ⊥r to it from the opposite ∠

2 Prove that if the middle points of the sides of a quadrilateral be joined, the figure formed is a parallelogram whose area = half that of the quadrilateral

3 Prove that if from a point two st lines be drawn to touch a ⊙, these st lines are equal (Text, p 180 Cor)

4 I. 22; 5 I 35, 6 II. 13, 7. III 32; 8 IV 5

1889

1 From a given point A, to draw a st line = to a given finite st line BC, the point A being in the line BC.

2 Divide the hypotenuse of a rt. ∠d Δ into 2 parts such that the difference between their squares shall be = the square on one of the sides

3 Prove that the rect under the sum and difference of two lines = the difference of the sqs on the lines (See II 5, Cor.)

4. Construct a square equal to a given equilateral Δ (II 14)

5 The three ⊥s let fall from the vertices on the opposite sides of any Δ, meet at point (Text, p 106)

6 I 44; 7. II. 11, 8 III 12; 8 IV. 5

1890

1 In a rt ∠d Δ, the line joining the rt ∠ to any pt except the middle point of the hypotenuse, is greater than one segment of the hypotenuse, and less than the other.

2. Prove that the area of a quard = area of a Δ, having 2 sides = the diagonals of the qadrl and the contained ∠ = that ∠ between the diagonals

3 In the fig. of II 11, prove that the rect. contained by the two parts = the difference of the sqs on the two parts

4 Show that two, and only two, tangents can be drawn to a ○ from a given point outside it (Text, p 180 Cor)

5. If two opposite sides of a quadrilateral, inscribed in a ○ are equal, prove that the other two sides are parallel

6 P is a point in APB, an arc of a ○, the tangent at P meets the chord AB produced in R, and AQ, ⊥ to AB, in Q, and QR is bisected in P Prove that ∠ ABP=2 ∠ BAP.

7 I 6, 8 I 32, 9 II 11, 10 III 17, 11 III 28 12 IV 10

1891

1 Define —A plane angle, centre of a ○ parallel str lines, ∠ *of* a segment, ∠ *in* a segment

2—4 I 21, I 47 II 13

5 Let B and C be the two fixed pts and PQ a str line in the same plane as BC Find the position of the pt A on the st line PQ, which is such that the sum of the squares on AB, AC is *least*

6—7 III 3, III 32

8 Draw a *common tangent* to two ○s, and shew that 4 common tangents may be drawn to the given ○s (Text p 218, Ex 17)

9 Give only the construction of —IV 4, IV. 10

10 In the Δ ABC O is the centre of the inscribed ○, and O_1, O_2, O_3 the centres of the *escribed* ○s Show that the 4 ○s each of which passes through 3 of the points O_1, O_2, O_3 are all equal

1892

1 Define —A plane surface, a rhombus, and an axiom What axiom affords the ultimate test of equality of two geometrical magnitudes?

2 I 7, I 23, 4 II 1, 5 III 1, 6 III 10, 7 IV 16

8 Show that, if a polygon inscribed in a ○ be equilateral, it is also equiangular (Text p 275, Ex 2)

9 Bisect a quadrilateral figure, by a straight line, drawn through an angular point (Text p 177, Ex 39)

10. Describe a ○ to touch a given ○, and also to touch a given straight line at a given point (Text p 221, Ex 36)

11 Prove that of all Δs of given base and area, the isosceles is that, which has the *least perimeter*

1893

1 Define —A rt ∠, a rectangle, a tangent to a ☉, and a regular polygon

2. I. 4; 3. I. 43; 4 II. 12, 5 III. 20; 6 III 36, 7 IV. 2

8. Bisect a △ by a straight line, drawn through a given point in one of its sides (174)

9 Two ☉s touch each other externally in A, and a straight line touches them in B and C respectively Prove that the ∠BAC is a ∠.

10 Given the base, vertical ∠ of a △; find the *locus* of the centre of the inscribed ☉. (Text p. 228, Ex 36)

1894.

1 I. 2. 2; I 27, 3 II 14; 4 III. 2; 5. III 24; 6 III 34

7. By the Fourth Book of Euclid you are required to construct an ∠ = to the *one-thirtieth* part of a rt ∠

8 Trisect a rt ∠

9. Describe a ☉ passing through two given points and touching a given straight line. (Text p 235, Ex 21.)

10 Given two pts. A and B and a st line L; find a pt. P in L such that $AP + BP$ shall be a *minimum* (Text p 243, Ex 3)

1895.

1. On the same base and on the same side of it, there cannot be two different *equilateral* △s constructed; prove this, and quote accurately the enunciation of the proposition of which this is a *particular* case. (A Case of I 7)

2 Being given two straight lines meeting in a point, you are required to draw through the point a straight line which shall make with one of the straight lines an ∠ = to the ∠ it makes with the other (I 9)

3 If two squares are equal, their sides are also equal · prove this State and prove the proposition in the First Book of Euclid in which the preliminary portion of this question is used. (See Text p 85, *obs*; and I. 48)

4. Being given a ☉, you are required to find a point such that all straight lines drawn from it, to the ☉ce are equal (III. 1.)

5 Being given a ☉, and a point *outside* it, you are required to draw a straight line through the point to meet the

○, and which on being further produced in the same direction will not cut the ○ (III 17, Case 2nd)

6 To a given straight line, you are required to *apply* a parallelogram, = in area to a given *equilateral* Δ, and containing an ∠ = to an ∠ of an equilateral Δ (A Case of I 44)

7 Being given a Δ you are required to find a point equidistant from the *three* vertices quote the enunciation of the proposition of which this is a part (IV 5)

8 Being given a Δ, find a point such that the ⊥s from it on the sides are equal, quote the enunciation of the proposition of which this is a part (IV 4)

9 The ⊥s dropped from the vertices on the opposite sides of a Δ are *concurrent* (Text p 224, Ex 19)

10 Being given two intersecting lines and a point O, you are required to draw through O a straight line meeting the given lines in P and Q so that the rectangle OP OQ may be given

1896

1. Enunciate and prove I 16

The side CA of the Δ ABC is produced to D, prove that the ∠ BAD is greater than the ∠ ACB

2 I 48

3 II 14

4 III 4

5 Enunciate and prove III 31

6 IV 4

7 From the ends of the base of a Δ, ⊥s are drawn to the bisector of the vertical ∠, prove that the feet of the ⊥s are equally distant from the middle point of the base.

8 Find the *locus* of a point such that the sum of the square on its distances from two given points may be = to a given square

9 In any Δ, if the ⊥s drawn from the vertices on the opposite sides are produced to meet the circumscribed ○, then each side bisects that portion of the line ⊥ to it, which lies between the *orthocentre* and the ○ce (See Ex 21, p. 226)

10 From an external point P, two tangents are drawn to a given ○ whose centre is O, and OP meets the chord of contact at Q, if R be the middle point of PQ, prove that RP is = to a tangent from R to the given ○ (See p. 233, Text.)

1897.

1 Prove I. 20, giving the construction for each of the *three* cases

Prove that the sum of the two sides of a Δ, is greater than twice the straight line drawn from the vertex to the middle point of the base

2. Prove I. 47

Give a proof of I 47 by showing how two squares may be cut into pieces and put together so as to form a third square.

3 II 11.

In what proposition in the first *four* Books of Euclid, is this proposition used? Explain briefly how the construction enables us to describe a regular *pentagon* on a given line (See IV. 10)

4 III. 12

In the enunciation of III. 12, is it strictly correct to speak of *the* point of contact?

5. Prove III. 36

6 Describe a ○ which shall touch a given straight line and pass through two given points (Ex 21, p 235, Text)

7 Prove IV 5, giving the figures for all the cases that may arise, and showing from the construction that the ⊥s at the middle points of the sides of a Δ, meet at the same point

With the help of the *rider* above mentioned, prove that the three ⊥s of a Δ, drawn from the vertices to the opposite sides, meet at the same point

8. Why does Euclid define a *point* as having no *magnitude* and a *straight line* as having no *breadth*?

9 Write a short *essay* on Euclid's *theory of parallel straight lines*

10 What are *converse* propositions? Enumerate all the instances of converse propositions in the first *four* Books of Euclid How does Euclid generally prove converse propositions? Do you know of any exceptions of this general rule?

CAMBRIDGE UNIVERSITY PAPERS.

MATHEMATICAL TRIPOS

Riders only

1848

1 If the two diagonals be drawn, shew that a parallelogram will be divided into four equal parts In what case will the diagonal bisect the ∠s of the parallelogram?

2 Shew that all equal straight lines in a ○, will be touched by another ○

3 If two straight lines AEB, CED in a ○ intersect in E, the ∠s subtended by AC and BD at the centre, are together double of the ∠ AEC

1849

1 Describe on a given finite straight line, an isosceles Δ, the sides of which shall be each = to twice the base

2 In Euclid's fig of II 11, shew that *four* others lines besides the *given line*, are divided in *medial section*

3 Describe a ○ touching one side of a Δ, and the produced parts of the other two (Text, p 255)

1850

1 If the opposite sides, or the opposite ∠s, of any quadrilateral figure be equal, or if its diagonals bisect each other, the quadrilateral is a parallelogram.

2 Given a square, and one side of a rectangle, which is = to the square, find the other sides

3 The *greatest* rectangle that can be inscribed in a ○, is a *square*

4. Divide a ○ into two segments, such that the ∠ in one of them, shall be *five* times in the other

5 Shew that the base of the Δ in IV 10, is = to the side of a regular *pentagon* inscribed in the smaller ⊙ of the figure.

1851

1 ABC, ACD be two equal Δs, upon the same base AB and on the opposite sides of it Join CD, meeting AB in E, shew that CE = ED

2 If ABC be a $\triangle$ whose $\angle$ A is a rt $\angle$, and BE, CF be drawn bisecting the opposite sides respectively, shew that $4(BE^2+CF^2)=5\ BC^2$

3. If a polygon of an *even* number of sides be inscribed in a $\bigcirc$, the sum of the alternate $\angle$s, together with two rt. $\angle$s, is =to as many rt $\angle$s as the figure has sides

4 In a given $\bigcirc$, inscribe a $\triangle$, whose $\angle$s are as the numbers 2, 5 and 8

1852

1 Divide a $\triangle$ by two straight lines, into *three* parts, which when properly arranged, shall form a parallelogram, whose $\angle$s are of given magnitude

2 Triangles are described on the same base and having the difference of the squares on the other sides constant; shew that the vertex of any $\triangle$ is in one or other of two *fixed* straight lines.

3 Two equilateral $\triangle$s are described about the same $\bigcirc$, shew that their intersections will form a *hexagon* (equilateral), but not generally (equiangular)

1853

1. If lines be drawn through the extremities of the base of an isosceles $\triangle$, making $\angle$s with it, on the side *remote from the vertex*, each=to one-third of one of the equal $\angle$s, and meeting the sides produced; prove that *three* of the $\triangle$s thus formed, are isosceles

2 Through two given pts draw two st lines, forming with a st line given in position, an equilateral $\triangle$

3 In the fig of II 11, if H be the point of division of the given line AB, and DA be the side of the square which is bisected in E, and if DH be produced to meet BF in L, prove that DL is $\perp$ to BF, and is divided by BE in *medial* section

4 Through a given pt *without* a $\bigcirc$, draw a chord, such that the difference of the $\angle$s in the two segments into which it divides the $\bigcirc$, may be=to a given $\angle$.

5 With a given pt as centre, describe a $\bigcirc$, cutting a given line in two pts so that the rect. contained by their distances from a fixed point in the line, may be=to a given square

1854.

1 If K be the common angular pt of the parallelograms about the diameter, and BD the other diameter, the difference of the parallelograms, is=2ce $\triangle$ BKD

2 Produce a given st line, to a pt such that the rect. contained by the whole line thus produced and the part produced shall be = to the sqr on the given st line

3 If the opposite sides of the quadl. be produced to meet in P, Q, and about the Δs so formed *without* the quadrilateral, ⊙s be described meeting again in R, shew that P, R, Q will be in one st line.

4 On a given st line, as base, describe an isosceles Δ having the *third* ∠ = *treble* of each of the ∠s at the base (See notes on IV 10)

1855

1 Prove that the sum of the distances of any point from the 3 ∠s of a Δ, is greater than half the perimeter of the Δ

2 If a st line be drawn parallel to the hypotenuse of a rt ∠d Δ, and each of the acute ∠s be joined with the pts, where this line intersects the side respectively opposite to them, the squares on the joining lines are together = to the squares on the hypotenuse and on the line drawn parallel to it

3 Divide a given st line into two parts, such that the square on one of them, may be = 2ce the sq on the other (Text p 85, Ex 2)

4 If any number of Δs, upon the same base BC, and on the same side of it, have their vertical ∠s equal, and ⊥s meeting in D, be drawn from B, C upon the opposite sides, find the *locus* of D, and shew that all the lines which bisect the ∠ BDC, pass through the same pt

5 If the ⊙ inscribed in a Δ ABC, touch the sides AB, AC in the pts D, E, and a st line be drawn from A to the centre of the ⊙, meeting the ⊙ce in G, shew that G is the centre of the ⊙ inscribed in the Δ ADE

1856

1 Of all parallelograms, which can be formed with diameters of given length, the *rhombus is the greatest*

2 If AB, (one of the equal sides of an isosceles Δ ABC), be produced beyond the base to D, so that BD = AB, shew that $CD^2 = AB^2 + 2\ BC^2$.

3 Shew how to derive a *hexagon* from an *equilateral* Δ inscribed in a ⊙, and from this construction, shew that a side of the *hexagon* equals the radius of the ⊙; and that the *hexagon* is double of the Δ.

1857

1 ABC is an isosceles △, of which A is the vertex: AB, AC are bisected in D and E respectively; BE, CD intersect in F, shew that the △ ADE is = to *three* times the △ DEF

2 The base of a △ is given, and is bisected by the centre of a given ⊙, the ⊙ce of which is the *locus* of the vertex, prove that the sum of the squares on the two sides of the △, is *invariable*

3 Prove that the sum of the ∠s in the *four* segments of the ⊙, *exterior* to the quadrilateral, is = to *six right angles*

4 Circles are inscribed in the two △s formed by drawing a ⊥ from an ∠ of a △ upon the opposite side, and analogous ⊙s are described in relation to the two other like ⊥s: prove that the sum of the diameters of the *six* ⊙s together with the sum of the sides of the *original* △ is = to twice the sum of the three ⊥s

1858

1. Assuming as an axiom that "two st lines cannot both be parallel to the same straight line," deduce Euclid's 12th axiom as a Corollary of I. 28

2 Produce a given st line, so that the sum of the squares on the given st line, and the part produced, may be = to twice the rectangle contained by the whole line thus produced, and the part produced.

3 Describe a ⊙ which shall touch a given straight line, at a given point, and bisect the ⊙ce of a given ⊙.

1859

1 *Trisect* a parallelogram by st lines drawn from one of its angular points.

2. Prove that, in any quadrilateral, the squares on the diagonals are together = to twice the sum of the squares on the straight lines joining the middle points of the opposite sides

3 Two equal ⊙s touch each other *externally*, and through the point of contact, chords are drawn, one to each ⊙, at rt ∠s to each other prove that the st. line, joining the other extremities of these chords, is = and parallel to the st line joining the centres of the ⊙s.

4. Triangles are constructed on the same base, with equal vertical ∠s, prove that the *locus* of the centres of *escribed* ⊙s each of which touches one of the sides *externally*, and the other side

and base produced, is an arc of a ○, the centre of which is on the ○ce of the ○, circumscribing the Δ

1860

1 If a st line DME be drawn through the middle point M of the base BC of a Δ ABC, so as to cut off equal parts AD, AE from the sides AB, AC, produced if necessary, respectively, the shall BD = CE

2 Shew how to construct a rectangle which shall be = to a given square, (1) when the *sum*, and (2) when the *difference* of two adjacent sides, is given

3 If two chords AB, AC be drawn from any point A of a ○, and be produced to D and E, so that the rectangle AC AE is = to the rectangle AB AD, then, if O be the centre of the ○, AO is ⊥ to DE

4 If A be the vertex, and BD the base of the constructed Δ in IV 11, being one of the points of intersection of the two ○s employed in the construction, and E the other, and AE be drawn meeting BD produced in F, prove that FAB is another isosceles Δ of the same kind

1861

1 If ABC be a Δ, in which C is a rt ∠, show how, by means of Book I, to draw a st line parallel to a given straight line, so as to be terminated by CA and CB, and bisected by AB

2 If ABC be a Δ, in which C is a rt ∠, and DE be drawn from a point D in AC, at rt ∠s to AB, prove, without using Book III, that the rect AB AE = rect AC AD

3 Two ○s intersect in A and B, and CBD is drawn ⊥ to AB, to meet the ○s in C and D, if AEF bisect either the *interior* or *exterior* ∠ between CA and DA, prove that the tangents to the ○ at E and F, intersect in a point on AB produced.

4 Describe a ○ touching the side BC of the Δ ABC, and the other two sides prouced, and prove that the distance between the points of contact of the side BC, with the inscribed ○, and the latter ○, is = to the difference between the sides AB and AC

1862

1 Upon the sides AB, BC, and CD of a parallelogram ABCD, *three equilateral* Δs are described, that on BC towards the *same parts as the parallelogram*, and those on AB, CD towards the *oppo-*

site parts Prove that the distances of the vertices of the Δs on AB, CD, from that on BC, are respectively = to the two diagonals of the parallelogram

2 Divide a given st line into two parts, so that the squares on the whole line, and on one of the parts, may be together *double* of the square on the other part

3 A Δ is *turned about its vertex*, until one of the sides intersecting in that vertex is in the same st line as the other previously was, prove that the line, joining the vertex with the point of intersection of the two positions of the base, produced if necessary, bisects the $\angle$ between these two positions

4 Prove that the *smaller of the two* $\odot$s, employed in Euclid's construction of IV 10, is = to the $\bigcirc$ described about the required Δ

1863

1. Two Δs ABC, A′B′C′ have their sides respectively parallel BB_1, CC_1 are drawn $\perp$ to B′C′, CC_2, AA_2, to C′A′, and AA_3, BB_3, to A′B′ Prove that the sum of the sqs on AB_1, BC_2, CA_3 together, is = to the sum of those on AC_1, BA_2, CB_3 together

2 Divide a given st line into two parts, such that the rectangle contained by the whole, and one part may be = to that contained by the other part and a given st line

3 Two equal $\bigcirc$s intersect in A, B, PQT a $\perp$ to AB meets it in T, and the $\bigcirc$s in P, Q, AP, BQ meet in R, AQ, BP in S, prove that the $\angle$ RTS is bisected by TP

1864.

1 If a quadrilateral figure have two sides parallel, and the *parallel sides be bisected*, the line joining the points of bisection shall pass through the point in which the diagonals cut one another

2 Divide a given st line (when possible) into *three* parts such that the rectangle by two of them shall be = to a given rectilineal figure, and that the squares on these two parts shall together be = to the square on the *third*

3 If from a given point A without a given $\bigcirc$, any two st lines APQ, ARS be drawn, making equal $\angle$s with the diameter which passes through A, and cutting the $\bigcirc$ in PQ and RS respectively, then PS, QR, shall cut one another in a given point

4 If a figure of any *odd* number of sides have all its angular points on the same $\bigcirc$, and all its $\angle$s equal, then shall its sides be equal

1865

1. Give a geometrical construction, for finding a point in a given straight line, the *difference* of the distances of which from two given points, on the same side of the line, shall be the *greatest* possible

2 The base BC of an isosceles △ABC, is produced to a point D, AD is joined, and in AD, a point E is taken, such that the rectangle AD AE, is = to the square on either of the equal sides AB, AC of the △, prove that the rectangle BD.CD is = to the rectangle AD ED.

3 A given st line is drawn at rt. ∠s to the st line joining the centres of two given ⊙s, prove that the difference between the squares on two tangents drawn, one to each ⊙, from any point on the given st line, is *constant*

4 Having given one side of a △, and the centre of the circumscribed ⊙, determine the *locus* of the centre of the inscribed ⊙

1866

1 Prove that a quadrilateral, which has two opposite sides and two opposite *obtuse* ∠s equal, is a parallelogram

2 Shew that the figure in ques 1st, is not necessarily a parallelogram, if the equal ∠s are *acute*

3 Prove II 9 by *superposition* of the squares or their *halves*

4 If *four* ⊙s be drawn, each passing through *three* out of *four* given points, the ∠ between the tangents at the intersection of two of the ⊙s, is = to the ∠ between the tangents at the intersection of the other two ⊙s

5 In a given ⊙, inscribe a △, such that, two of the sides of the △, shall pass through given points, and the third side at a given distance from the centre of the given ⊙

1867

1 Any two exterior ∠s of a △, are together greater than two rt ∠s

2 What is the *greatest value* which the *complements*, for a given parallelogram, in I 43, can have? (See Notes on I. 43)

3 Divide a given str line into two parts, such that the squares on the whole line, and on one of the parts, shall be together *double* of the square on the other part

4 If the chords which bisect two ∠s of a △, inscribed in a ⊙,

be equal, prove that either the ∠s are equal, or the third ∠ is = to the ∠ of an equilateral Δ

1868

1 OKBM and OLDN are parallelograms, about the diagonal of a parallelogram ABCD In MN, which is parallel to BA, take any point P, and prove that, if PC, produced if necessary, meet KL in Q, BP will be parallel to DQ

2 In a ΔABC—D, E, F are the middle points of the sides BC, CA, AB respectively, and K, L, M are the feet of the ⊥s *on the same sides from the opposite ∠s* Prove that the *greatest* of the rectangles contained by BC and DK, CA and EL, AB and FM, is = to the sum of the other two

3 Through a point within a ⊙, draw a chord, such that the rectangle contained by the whole chord and one part may be = to a given square

4 If two Δs ABC, A′B′C′ be inscribed in the same ⊙, so that AA′, BB′, CC′, meet in one point O, prove that, if O be the centre of the inscribed ⊙ of one of the Δs, it will be the centre of the ⊥s of the other

1869

1 ABC is a Δ, E and F are two points, if the sum of the Δs ABE and BCE be = to the sum of the Δs ABF and BCF then under certain conditions EF will be parallel to AC

Find these *conditions*, and determine when the *difference* instead of the *sum* of the Δs must be taken

2 Shew that in II 11, the point of section, lies between the extremities of the line

3 An *acute-angled* Δ, is inscribed in a ⊙, and the *paper is folded along each of the sides of the* Δ Shew that the ⊙ces of the three segments will pass through the same point

State the equivalent proposition for an *obtuse-angled* Δ

4 Shew that the ⊙s, each of which touches two sides of a *regular pentagon* at the extremities of a third, meet in a point

1870

1 ABCD is a square, and E a point in BC, a st line EF is drawn at rt ∠s to AE, and meets the st line, which bisects the ∠ between CD and BC produced in a point F; prove that AE is = to EF

2 The diagonals of a quadrilateral meet in E, and F is the middle point of the st line joining the middle points of the diagonals, prove that the *sum* of the squares on the str lines joining E to the angular points of the quadrilateral, is *greater* than the *sum* of the squares on the st lines joining F to the same points, by *four* times the square on EF

3 AB, CD are parallel diameters of two ⊙s, and AC cuts the ⊙s in P, Q, prove that the tangents to the ⊙s at P, Q are parallel

4 Describe an equilateral and equiangular *pentagon* about a ⊙, *without first inscribing one* (See Notes on IV 11)

1871

1 Through the angular points A, B, C, of a △, are drawn three parallel lines meeting the opposite sides in A′, B′, C′, respectively, prove that the △s AB′C′, BC′A′, CA′B′ are all equal

2 Produce a given line so that the square on the whole line thus produced, may be *double* the square on the part produced.

3 The opposite sides of a quadrilateral inscribed in a ⊙, are produced to meet in P, Q, and about the *four* △s thus formed, ⊙s are described, prove that the tangents to these ⊙s at P, Q, form a quadrilateral equal in all respects to the *original*, and that the line joining the centres of the ⊙s about the two quadrilaterals, bisects PQ

4 A △ is inscribed in a given ⊙, so as to have its centre of ⊥s at a given point, prove that the middle points of its sides lie on a *fixed* ⊙

1872

1 If CE, BD be the squares described upon the side AC, and the hypotenuse AB, and if EB, CD intersect in F—prove that AF bisects the ∠ EFD

2 Two ⊙s intersect in A, B, PAP′, QAQ′ are drawn equally inclined to AB to meet the ⊙s in P, P′, Q, Q′, prove that PP′ is = to QQ′

3 Having given an angular point of a △, the circumscribed ⊙, and the centre of the inscribed ⊙, construct the △

1875

1. A′, B′, C′ are the middle points of the sides of the △ ABC, and through A, B, C are drawn 3 parallel lines meeting B′C′, C′A′, A′B′, in *a*, *b*, *c* respectively prove that the △*abc* is ½ △ABC and that *bc* passes through A, *ca* through B, *ab* through C

2 If the diagonals AC, BD of the quadrilateral ABCD, (inscribed in a ☉, the centre of which is at O), intersect at rt ∠s in a fixed point P, prove that the feet of the ⊥s drawn from O and P to the sides of the quadrilateral, lie on a *fixed* ○, the centre of which is at the middle point of OP.

3 Through a *fixed* point O, any line OPQ is drawn cutting a *fixed* ○ in P and Q, and upon OP and OQ as chords, are described ☉s touching the *fixed* ○ at P and Q; prove that the two ○s so described, will intersect on another *fixed* ○.

4 Prove that the ○ drawn through the middle points of the sides of the Δs, in IV 10, will intercept portions of the equal sides such that a *regular pentagon* can be inscribed in the ☉, having these portions as two of its sides

1877

1 In a given Δ, inscribe a parallelogram = ½ the Δ, so that one side is in the same st line with one side of the Δ, and has one extremity at a given point of that side

2 If a line AB be bisected in C and produced to D, so that $AD^2 = 3\ CD^2$, and if CB be bisected in E, shew that $ED^2 = 3\ EB^2$.

3 If a quadrilateral be inscribed in a ○, and the middle points of the *arcs* subtended by its sides be joined, to make another quadrilateral, and so on, shew that these quadrilaterals *tend* to become *squares*

4 Prove that *four* ○s may be described, touching the *three* sides of a Δ, and that the square on the distance between the centres of any two, together with the square on the distance between the centres of other two, is = to the square on the diameter of the ☉ passing through the centres of any three

1893

1 I 4, 2 I 28, 3 I. 42, 4. II. 13, 5 III 6, 6. III 14 (*const*), 7 III 36

8 In a given str. line, find a point, equidistant from two given points

9 Draw a ○ of given radius to touch two given ○s

10 On the side AB of an equilateral Δ, a square ABDE is described, through A, the line AF is drawn parallel to BC to meet DE in F, and CA produced to meet DE in G Prove that AFG is an equilateral Δ

1894

1 I 5, 2 I 29, 3 I 49, 4 II 12, 5 III 5, 6 III 14, 7 III 37

2 Having given two sides of a Δ, and an ∠ opposite to one of them, construct the Δ

2 On the side AB of an equilateral Δ, a square ABDE is described Through A, the line AF is drawn parallel to BC to meet DE in F, shew that Δ AEF is $= \frac{1}{2}$ the equilateral Δ

MADRAS UNIVERSITY PAPERS

ENTRANCE EXAMINATION

GEOMETRY

1857

1 Define a straight line, a plane superficies, a circle, rhombus and parallel straight lines

2 Enunciate the postulates

3 I 5, 4 I 29

5 ABCD is a quadrilateral figure, the sides AB, DC are produced to meet in E and the sides AD, BC, to meet in F, shew that the ∠ BCD = sum of the ∠s at A, E & F

6 I 32, 7 II 11, 8 III. 17, 9 III 36

10 Describe a ◯, which shall touch a given line in a given point, and shall cut off from another given line, a chord of given length Shew that in general *two* ◯s may be drawn

1858

1 In every theorem, the hypothesis properly takes the form of a *sentence* commencing with particle Enunciate according to this form Props I 5 and I 7

2 The *shortest distance* between the two points is a st line, and that the shortest distance from a point to a st line is a ⊥ to the line (See Notes on I 20 and I 12)

3 I 21

4 Enumerate and prove the *properties* which belong to two parallel straight lines, intersected by another straight line

5 Interpret *arithmetically* the following *algebraic* formulæ — $(a+b)^2=a^2+2ab+b^2$, $(a-b)^2=a^2-2ab+b^2$, $(a^2-b^2)=(a+b)$

$(a-b)$, and state the corresponding *geometrical* propositions (Ans. II 4, II 7, II. 5.)

6 Give the construction of II 11.

7 II 14

8 In a ○ whose radius is "a" find the length of a chord whose shortest distance from the centre is "b"?

9 How is a line determined to be a tangent to a ○.

10 An ∠ is practically measured by the *graduated arc* of a ○, the angular point being the centre (*omit*).

(*a*) Applying the above method, prove III 20
(*b*) Prove that the ∠ formed by a tangent to a ○ and a chord through the point of contact, is measured by *half the intercepted arc*

11 III 22

12 Prove the *general* case of III. 35

1860.

1 Define an angle, right angle, acute angle, obtuse angle, right angled triangle, acute angled triangle and obtuse angled triangle How many angles can be formed at a point, by two, three, n lines?

2 Give the construction of I 3, not assuming those of the I 1 and I 2

3 (*a*) I 9

(*b*) If instead of the *ordinary* construction, the equilateral △ be drawn with its vertex *towards the given* ∠ in what case will the proof be unaltered? And when it is altered, what corresponding problem will be solved? (See Notes on I 9.)

4 I 26

5 Describe a parallelogram = to a given *hexagon*, and having an ∠ = to *one-third of a right* ∠

6. II 7

7 Prove either the II 12 or II 13 Enunciate the proposition of which, they (II 12 and 13) and I 47 are the *three* cases

8 Complete the ○, of which you have a given segment. (III 25)

9 III 34.

10 If ACB$=\frac{1}{n}$th of AB, shew that the diagonal of the square described upon AB, is = to $\sqrt{2}\, n$. AC

11 If any number of *concentric* ⊙s be cut by another ◯, the *common* chords shall be parallel

12 If from any point, equal tangents be drawn to any number of ◯s, these ◯s may all be intersected by another ⊙, which shall have two radii each of them for tangents

1861

1 Define a point, a line, a *plane* Can a *plane* be said to be made of lines, and a line made of points?

2 What is a postulate Why are postulates necessary? What are Euclid's three postulates?

3. (I 5 case 2nd)

4 I 11 What is the Corollary usually given to this proposition Do you know any *objection* to it?

6 Let two Δs stand upon a common base, either on the *same* or *opposite* sides, draw a line through the vertices and cutting the base, produced if necessary If the distance measured along this line, of the one vertex from the base, be *double* that of the other, the *area* of the 1st Δ is also *double* that of the 2nd Δ

7 II 8

8. Produce one of the sides of a square, so that the rectangle contained by the whole line thus produced, and the part of it produced, may be = to the given square

9. Define a segment of a circle, a sector of a circle and similar segments of circles

10 III 22, 11 III 35

12 If two tangents to a ◯, be drawn from the points A and B on the ◯ce, and intersect at C, and if AC be produced to meet the radius through B, at D, shew that the rectangle contained by AC and AD = the rectangle contained by BD and the *radius*

1862

1 Define a line, a circle, semi-circle, a polygon, an oblong, a rhombus and a trapezium

2 Write down the 9th, 10th and 11th axioms What is an axiom?

3 Shew upon what supposition, the construction of I 2 will pass into that of I 3 (Notes on I 2)

4 I 20

5 If two Δs have two sides of one, = to two sides of the other, each to each, and have also the $\angle$s *opposite* to one of the equal sides in each Δ, = to one another, and if the $\angle$s opposite to the other equal sides be both *acute* or both *obtuse*, then shall the third sides be equal, and also the remaining $\angle$s

6 I. 47

7. From AB, the side of an isosceles Δ (or from AB produced), cut off AD = AC the base, and from AC produced (or from AC) cut off AE = AB Let BC and DE meet in F, prove that DF = CF, and also that BF = EF

Shew also that DE and BC cannot be bisected in F

8 II 5, 9 II 14

10 If in a Δ ABC, the side CA exceed BC by *half* BA, shew that the square on CA is *greater* than the square on BC by the rectangle contained by BC and BA, together with the square on *half* BA.

11 III. 4, 12 III 26

13 If two $\bigcirc$s cut one another, and at a point of intersection, a tangent to the one, passes through the centre of the other, pass through the centre of the first

Under what conditions can the *four* radii drawn to the points of intersection, form a square?

1863

1 Define a point, a right angle, a circle, a rhomboid

2 I 1 How would you modify this, so as to describe an isosceles Δ with sides of given length?

3 (*a*) I 20

(*b*) The sum of the diagonals of a parallelogram, is *less* than the sum of the *four* lines which are drawn from any point to the $\angle$s of the parallelogram

4 I 31, 5. I 48, 6. I. 32; 7 II 5, 8 II. 14, 9 III 12, 10 III 31, 11 III 20

12 Two $\bigcirc$s of unequal radii, touch each other *externally*, and any line is drawn through the point of contact cutting both $\bigcirc$s Shew that the diameters drawn through the points of intersection, are parallel

1864

PART I

1. Define parallel straight lines, a circle, a *gnomon*, similar segments of circles.

2 I 21, 3 I 42.

4 On one of the sides of a △, describe an isosceles △, = to the *original* △

5 II 11

PART II

6 Two equal ○s cut one another in A and B The line joining their centres, cuts the ○s in C and D Shew that ABCD is a *rhombus* Can this rhombus under any circumstance become a square?

7 III 17, 8 III 22, 9 III 34

10 If two st lines AEB, CED in a ○, intersect in E, the ∠s subtended by AC and BD at the centre, are together = 2 ∠ AEC (Text p 222, Ex 1)

1865

1 Define a plane superficies, a right angle, and give the 12th axiom

2 I 12, 3 I 32 *Cor*

4 ABC is a given △, describe a △ PQR = to ABC, having a side = AB and a side PR double of AC

Shew that *two* such △s may be described

5 II 7, 6 II 14

7 ABC is an *acute* ∠d △ From the points A, B, C, ⊥s AD, BE, CF are drawn upon the opposite sides BC, CA, AB, shew that the squares on the three sides of the △ is double the sum of the rectangle contained by AB, AF, by BC BD, and by AC AF

8 III 13, 9 III 21, 10 III 32

11 If two ○s touch each other *internally*, the radius of the *larger* being double that of the *smaller*, prove that any chord of the *larger* ○ passing, through the point of contact, is bisected by the ○ce of the *smaller*

MADRAS MATRICULATION PAPERS

1866

1 Define —A plane, a semi-○, a trapezium

2 I 29

3 In the base AB of an isosceles △, any point P is taken. From Q and R the points of bisection of AP and BP, QD and

RG are drawn ⊥ to AB meeting the sides AC and BC respectively in D and G. Shew that AD=CG, and DC=BG.

4 I 36

5 Having given the 3∠s of a Δ, and the *difference* of any two of its sides construct the Δ.

6 II 6; 7 III. 17. 8 III 37.

9. Two ○s touch each other *internally* in A The diameter AC of the *larger* ⊙, meets the *smaller* ⊙ in B At B, a tangent is drawn meeting the larger ⊙ in D and E. Prove that CD is= to the tangent drawn from C to the smaller ⊙.

10. AB, AC are two tangents to a ○, meeting one another in A, and the ○ in B,C. Through B, a str line BD is drawn parallel to AC, meeting the ⊙ in D. Prove that BC^2 is=AB BD

1867.

1. I 15

2. I 33

3. ABCD is a parallelogram whose diagonals are AC, BD The side DA is produced to E, so that AE may be=AB, and EB is joined and produced to meet DC produced in F, and through F, a line FG is drawn parallel to CB meeting AB produced in G Prove that the diagonals of the parallelogram BCFG intersect at rt ∠s

4 I 48.

5 AB is a str line, and from A,B, ⊥s AC, BD, are drawn, such that AC+BD are=AB Prove that CD^2 is double of the sqs. on AC, BD

7 II 7; 8 III 32; 9 III 35

10 AEFB is a semi-○ whose diameter is AB Through E and F, the lines EG, FH are drawn ⊥ to AB meeting it in G and H A ○ is drawn touching the semi-○ *externally*, and the lines GE and HF produced If C be the centre of this ⊙, and CD be ⊥ from C on AB, prove that CD^2=AH BG.

11 AB is the diameter of a ⊙ ADBF, and C a point in AB produced, whence CD is drawn touching the ○ From centre C, another ⊙ is described cutting the ⊙ ADBF in D and F, and diameter in E If now a line be drawn through C ⊥ to CA, and any point P be taken in that line, prove that PE is=to the tangent drawn from P to the ○ ADBF.

1868

1 I. 20, 2 In any Δ CPY, prove that CP+CY are>PY.

3. The difference between any two sides of a Δ, is *less* than the 3rd side.

4 I 37

5 The *perimeter* of an isosceles Δ is > than that of an equal rectangle of the same altitude

6 I 32, with Corollaries

7 If the base XY of a Δ AXY be produced, and the exterior ∠s so formed be bisected, and if the interior and opposite ∠AXY so formed be bisected, prove that the bisecting lines will meet, and form an ∠ = ½ the other interior and opposite ∠A

8 If parallelograms AQPB, CPRD be constructed upon two of the sides QP and PR of any ΔPQR, and their sides AB and DC parallel to the sides of the Δ be produced to meet in a point E, if a straight line EP be drawn from that point to the vertex of the Δ, and if a parallelogram QFGR be constructed upon the base of the Δ, whose other sides are equal, and parallel to EP, then the parallelogram QFGR = parallelogram AQPB + parallelogram CPRD

9 II 13

10 PM is drawn ⊥ to the hypotenuse AB of a rt ∠d ΔABC, from any point P in AC Prove that the rect AP AC = the rect AM AB.

11 III 21

12 Through the points of intersection of two ○s, str lines are drawn intersecting, *within* one of the ○s, and *meeting one* of them again at A and B, and the *others* at C and D, shew that AB is parallel to CD

13 III 36

1871

1 Define —An isosceles Δ 2 I 5

3 If the str lines BD, CD bisecting ∠s B and C of a ΔABC, are = to one another, shew that the Δ is isosceles

4 I 32, part 2nd

5 The difference of the ∠s at the base of any Δ, is *double* of the ∠ contained by the line drawn from the vertex ⊥ to the base, and the line bisecting the ∠ at the vertex

6 II 13

7 ABC is a Δ having ∠B = ∠C, and BD is ⊥ to AC, shew that $BC^2 = 2AC\ CD$

8 III 11 What is assumed, when it is said, that the line joining the centres, shall pass through the point of contact

9 If in any two ○s that touch, there be drawn parallel diameters, an extremity of each diameter, and the point of contact, shall be in the same str line

10 III. 15

11 Through a given point within a ◯, draw the *least* chord

12. (*a*) III 17 with *Cor* (*b*) From a point, *two* and only two, tangents can be drawn to a ◯

13 III 32

14 If two ◯s touch each other, any str line through the point of contact, shall cut off similar segments from the two ◯s

1874

1 Define —A centre of a ◯, an acute ∠d Δ, a complement of a parallelogram, and a gnomon

2 I 14

3 Two parallelograms ABCD and CFGH having the ∠BCD in one=the ∠FCH in the other, are placed so that the side DC is in the same str line with the side CF, shew that BC is in the same str line with CH

4 What mode of proof is used in the I 14? When does Euclid chiefly use it? What other examples are there of its use in Book I

5 I 37

6 ABC is a Δ, BD is drawn from B to meet CA produced in D, AE is drawn from A, parallel to BD, meeting BC in E, and CF is drawn parallel to BD or AE from C meeting BA produced in F, shew that ΔBEF = ΔCED

7 II 7; 8 (*a*) II 14 (*b*) Deduce from the figure, a method of applying to a given str line a rectangle=to a given square

9 III 26, 10 III. 32, state the converse of III 32

11 From the middle point of the side AB of an isosceles ΔABC, a str line is drawn at rt. ∠s to it, meeting the base BC produced at D, shew that a ◯ drawn through the points A,C,D will touch AB

1878.

1. I. 23

2 An obtuse ∠ABC is divided by BD, so that ∠CBD is =2∠ABD, CD is parallel to AB and CE at right ∠s to CD, meets DB produced in E, shew that DE=2BC

3 I 47

4 In the figure of I 47, the *area* of a *six-sided* figure formed by a side of each square and the 3 str lines which join the adjacent corners of the squares is=4 times the area of the original Δ + twice the square on the hypotenuse

5 Two equal ⊙s intersect one another, so that the centre of each ⊙ is on the ⊙ce of the other From a point on the ⊙ce of one of these, str lines are drawn through the centres A and B, meeting the ⊙ce of the other ⊙ in P and Q If AQ and PB be joined, shew that one of the ∠s APB, AQB is *four times* the other

6 II. 6, 7 III 37

9 Produce a given str line, so that the rectangle contained by the whole line thus produced, and the part produced may be = to a given square

10 ACB, DCB are two ⊙s, intersecting in B, C, P is a point in BC produced PA is a tangent to ACB, PDE a chord of DBC AD and AE cut the ⊙ ACB in F and G Shew that FG is parallel to DE

1882

1 Define —An acute ∠d △, a rhomboid, and a rectangle

2 I 10

3. Determine the positions of all points that are equidistant from two given points

4 I 34

5 If both diagonals of a quadrilateral bisect the figure; shew that it is a parallelogram

How is it said to be contained ?

6 II 3

7 From B in the △ABC, BD is dropped ⊥ to AC and falling within the △, shew that $AB^2 + AC \cdot CD = BC^2 + AC \cdot AD$

8 III 3

9 Shew that the str line that bisects two parallel chords of a ⊙, passes through the centre

10 III 22

11 ABC is a △ inscribed in a ⊙, and any str line DE is drawn parallel to the tangent at A, and cutting the sides AB, AC in D, E respectively, shew that a ⊙ will pass through the points B, C, E, D

1889

1 Define —A rectangle, square, rhombus, I. 26

2 Describe *rhombus* = to a given rectangle

3 A square and a rhombus stand on the same base, which has a larger *area*, and why ?

4. I 41.

5 E is a point in the side AD of a parallelogram ABCD, prove that $\Delta EAC + \Delta EBD = \frac{1}{2}$ the parallelogram ABCD

6 II 5.

7. If X, Y be the lengths of two str lines, of which the *first is greater*, illustrate by a figure the *formula* $(x+y)^2=(x-y)^2+4xy$

8 Constuction of III. 17

9. AB is a diameter of a ◯, which intersects at C and D any ◯ having A for its centre Prove that BC, BD are tangents to the *2nd* ◯

10 III 20

11 ABCD is a square, arcs of ◯s AEC and BED are described with centres B and C respectively, cutting one another at the point E, within the square Prove that the ∠ BDE = ⅓ of a *right* ∠

1890

1 I 24, 2 Give the constructions of I. 44 and II 14

3 II 13 4 III 21

5 ABP is a Δ described on a given base AB, what is the *locus* of P when the vertical ∠ is of given magnitude? For what position of P will the Δ be *greatest*

6 (*a*) Distinguish between a *segment* and a *sector* of a ◯ (*b*) Prove that any str line drawn through the middle point of the line joining the centres of two equal ◯s, divides each of the ◯s into segments, such that segments of the one ◯ are = to the segments of the other

7 (*a*) III 32, (*b*) Enunciate the *converse* of III 32

8 Prove that the quadrilateral formed by the tangents at the extremities of any two diameters of a ◯, is generally *a rhombus*

BOMBAY UNIVERSITY PAPERS

MATRICULATION EXAMINATION

Geometry

1859

1 I [illegible] I 13, 3 I 32, 4 I 34, 5 II 1, 6 II [illegible] III 22, 10 IV 4, 11 IV 10

12 Show how to *trisect* a right ∠

13 Prove that the two diagonals of a parallelogram, bisect each other

14 Two ⊙s intersect in the points P and Q, show that the line joining the centres of the two ⊙s, cuts PQ at rt ∠s

15 Let the opposite sides of a quadrilateral, be produced to meet in P and D Two Δs will thus be formed *lying outside* the quadrilateral Let the ⊙s circumscribing these Δs, intersect in R Prove that P, R, Q, are in the same st line

16 How many *degrees* are there in the vertical ∠ of the Δ formed in IV 10 (Ans 36 degrees)

1860

1 I 6, 2 I 16, 3 I 42, 4 I 48, 5 II 5, 6 III 13, 7 III [illegible] 8 IV 14

9 Prove, that if ABCDEF be any *hexagon* inscribed in a ⊙ the ∠s A C E are together = to *four* right ∠s

10 If two diagonals of an equilateral and equiangular *pentagon* cut one another, shew that each of the greater segments is = to a side of the *pentagon*

1861

1 I 5, 2 I 7, 3 II 9, 4 II 12, 5 III 32, 6 IV 4, 7 IV 10

8 Cut off by a line parallel to the base of a Δ, a part which shall be *nine tenths* of the Δ

1862

1 I 32, 2 II 14, 3 III 33, 4 IV 11

5 Of Δs, which have the same vertical ∠, and whose bases pass through the same point, the *least* is that whose base is *bisected* by this point

6 The vertical ∠ of a Δ, is *equal* to, *less* than or *greater* than, a right ∠, according as half the base is *equal* to, *less* than or *greater* than the line joining the vertex with the point of bisection of the base

7 The vertical ∠ of a Δ, differs from a right ∠ by the ∠ between the base and the diameter of the circumscribing ⊙.

8 The square on the side of an equilateral Δ, is *triple* the square on the radius of the circumscribing ⊙.

9 State the subjects treated of in the *1st*, *2nd*, *3rd*, and *4th* Books of Euclid's Elements

1863 March

1 I 9, 2. I 16, 3. I 32; 4 I. 37; 5 II 10, 6 II 11; 7 III 3, 8 III. 20, 9 III. 21, 10 III. 33; 11. IV. 5, 12 IV. 10.

13 Describe a Δ, that shall be = to given *pentagon*

14 Upon a given base, describe an isosceles Δ whose vertical ∠ shall be = *one-half* of the vertical ∠ of a given isosceles Δ upon the same base.

15 Cut off from a given ⊙, a segment that shall contain an ∠ of 60 degrees (A case of III 34.)

1863 November

1 I 23, 2 I 34, 3 II 4, 4 II 13, 5. III 1, 6 III. 21; 7. IV 13, 8. IV 15.

9 Through a given point, between two lines that meet, draw a str line terminated by those two lines, which shall be bisected at the given point.

10. If from any point within a parallelogram, lines be drawn to the *four* ∠s of a parallelogram each pair of opposite Δs thus formed will be together = to *half* the parallelogram.

11 In any Δ, the squares on the two sides are together *double* of the squares of *half* the base, and of the line drawn from the vertex to the middle point of the base

This *Ex* is very *important* (See Text Ex 24, p. 147)

12 If two str. lines AEB, CBD, in a ⊙, intersect in E, the ∠s subtended by AC and BD at the centre, are together *double* of the ∠ AEC (See Text p 222, Ex 1.)

13 Describe a ⊙ touching one side of a Δ, and the other two sides produced. (See Text, p 255.)

1864.

1 I 2, 2 I 24, 3 I 35, 4 II 6, 5 III 21, 6 III. 35, 7 III. 36 Cor, 8 IV. 10, 9 IV. 16

10 From point A on the line BC (and between the extremities B and C), draw the line AD through B, = to BC

11 If from a point A, *outside* a parallelogram BCDE, the st lines AB, AC, AD, and AE be drawn, then the *difference* between the Δs ABC and AED, is = to *half* the parallelogram

12 Given the base, the altitude, and the vertical ∠ of a Δ, construct it

13 Prove that the bisectors of the three ∠s of a Δ, meet in a point (See Text p 103, Ex 2)

1865.

1 I 12, 2 I 21, 3 I 32, 4 II 6, 5 II 14, 6. III 12, 7 III. 22, 8 IV 2

9 Draw two *concentric* ○s, such that, those chords of the *outer* ○ which touch the *inner*, may be = to its diameter

10 Inscribe a ○ in a given *rhombus*

1866

1 I 20, 2 II. 11, 3 III 22, 4 III 35, 5 IV 2, 6 IV 13

7 If two sides of a Δ be bisected, the str line which joins the point of section, is parallel to the base and = to one *half* of it (See Text p 96-97 Ex 2-3)

8 The sum of the squares on the diagonals of any parallelogram, is = to the sum of the squares on the sides of the parallelogram

9 The square inscribed in a ○ is = to *half* the square described about it (See Notes on IV 6 and 7)

1867.

1 I 21, 2 I 44, 3 II 14, 4 III 34, 5 IV 3, 6 IV 7

7 If in Euc I 1, an equilateral Δ be also described on the other side of the given line, what figure will the two Δs form? What parts of a Δ must be given, in order that the Δ may be described? (See notes on I 1)

8 What conditions must be fulfilled, that a ○ may pass through *four* given points

6 If a side of an equilateral △ be *six inches*, what is the length of the *radius* of the *inscribed* ⊙.

1868

1 I. 11; 2 I. 42, 3 I. 47; 4 II. 7; 5. II. 11; 6 III 17, 7 IV 5, 8 IV. 10

9. Prove that two of the diameters of the *smaller* squares are in a str line, and the other two are parallel

1869

1 I 20, 2 I 42, 3 II 4, 4 III 16; 5 III 31, 6 IV 4

7 In a given indefinite st. line, find a point such that the sum of its distances from two given points, on the same side of the st line, shall be the *least* possible (Text p 243, Ex 3)

8. When is the rectangle contained by the two parts, the *greatest* possible, and the sum of the squares on the two parts the *least* possible?

9 Divide a given line into two such parts that the sum of the squares on the two parts equals a given square

10 Show that, if the ⊂ce of a ⊂, passes through *three* angular points of a *regular polygon*, it will pass through all of them.

11. In a given ⊙, inscribe a △, whose ∠s shall be as 1 2 3

1870

1 I 12, 2 I 34, 3 II 5, 4 II 14, 5 III. 22, 6. III 35, 7 IV. 3, 8 IV 6

9 If the diameters of a parallelogram are at rt ∠s to one another, the sides are all equal.

1871

1 I 27, 2 I 46, 3 II 2, 4. II 13, 5 III. 12, 6. III 34, 7 IV 5, 8. IV 10

9 Describe *three* ⊂s with radii in proportion to 1 2 3, so that they touch one another

1872

1 I. 8, 2 II 11, 3 III 20, 4. III 25, 5 IV. 2, 6 IV 5.

7 To a given str line, apply a Δ, which shall be=to a given Δ and have one of its ∠s=to one ∠ of the given Δ.

8 Show that the *perimeter* of an isosceles Δ is *less* than that of any other Δ of *equal* area, standing on the same base

1873

1 I 1?, 2 I 48, 3 II 13, 4 III 17, 5 III 37, 6 IV ?

7 In the diameter of a ○ produced, determine a point so that the tangent drawn from it, to the ○ce, shall be of given length

1874

1 I ?1, 2 I 34, 3 II 7, 4 III 23, 5 III 35; 6 IV 3

7 Upon a given base, describe an isosceles Δ, having each of the equal sides to a given str line What *limitation* is necessary with respect to length of the given str line, in order that the Δ may be possible?

8 Show that the *two* tangents can be drawn to a ○ from a given *external* p and that they are equal (See Text p 180, Cor)

9 Show that if the str lines joining the centres of the *inscribed* and *circumscribed* ○s of a Δ, passes through one of its angular points, the Δ is *isosceles*

1875

1 I 14, ? I 34, 3 II 10, 4 III 20, 5 III. 36, 6 IV 5, IV 10

8 Construct an isosceles Δ, which shall have *one-third* of each ∠ at the base to *half* the vertical ∠

9 Bisect a parallelogram by a str line drawn through a given point within it

10 If from any point without a ○, str lines be drawn touching it, the ∠ contained by the tangents is double the ∠ contained by the str line joining the points of contact and the diameter drawn through one of them.

1876

1 I 22, 2 II 14, 3 IV 9, 4 IV 15

5 Construct a right ∠d Δ, having given the *hypotenuse* and the *difference* of the sides

9. In a Δ, whose vertical $\angle$ is a right $\angle$, a str. line is drawn from the vertex $\perp$ to the base; show that the square on this $\perp$ is = to the rectangle contained by the segments of the base

7 On a given str line as base, describe an isosceles Δ, having the *third* $\angle$ = *treble* of each of the $\angle$s at the base (See IV 10, ACD is the Δ reqd)

1877

1. I 20, 2 I 32 Cor., 3 I 48; 4 II. 9, 5 III 13, 6 III. 32; 7 IV 4, 8 IV 15

9. The opposite $\angle$s of a quadrilateral figure are together = to *two right angles*; show that it may be inscribed in a $\odot$ (Converse of III 22, See Text, p 189.)

10 AB, AC are two chords of a $\bigcirc$, and BD is drawn parallel to the tangent at A to meet AC in D, prove that the $\odot$ described round the Δ BCD, touches AB.

11 Show that, the diameter of a $\odot$ inscribed in a right angled isosceles Δ is = to the difference between the hypotenuse and the sum of the other two sides

1878

1 I 13; 2 I 34, 3 II 5, 4 II 14, 5 III 24, 6 III. 35, 7. IV 5

8 Bisect a Δ by a str line drawn from a given point in one of its sides (Text p 113, Ex 36)

9 AB is a diameter of a $\bigcirc$, C any pt. in its $\bigcirc$ce, AC, BC, produced meet the tangents at B and A in D and E, and the tangent at C meets the same tangents in F, G Show that FG is *half* of BD + AE

1879

1 I 43; 2 II. 14, 3 III 17, 4 IV 4; 5. IV 11

6 ABCD is a parallelogram, a str line EF is drawn parallel to the diagonal AC, meet AD, DC or those produced in E, F, respectively show that the ΔABE is = to the ΔBCF

1880

1 I 4, 8, 26, 2 I. 34

3. Two isosceles Δs stand on *opposite* sides of the *same* base, show that the str line joining their vertices, bisects their *common base* at right $\angle$s

4 Show that the diagonals of a *rhombus*, bisect each other at right ∠s

5 Two equal ◯s touch each other *externally*, and through the point of contact, chords are drawn to each ◯ at right ∠s to each other, prove that the str line joining the other extremities of these chords, is = and parallel to the str. line joining the centres of the ◯s

6 III 36

Hence show that if two ◯s intersect each other, their common tangent is bisected by the line joining the points of intersection produced

7 IV 5.

Hence prove that, if the centre of this ⊙ coincides with that of the inscribed ◯, the Δ is equilateral

1881.

1 I 35, 2 II 11, 3 II 2, 4 III 28, 5 IV. 16

6 AD, CE be drawn ⊥ to the sides BC, AB of the ΔABC, and DE be joined Prove that the ∠ADE, = ∠ACE

7. In a given ◯, inscribe a Δ, whose ∠s shall be as $\frac{1}{6}$ $\frac{1}{3}$ $\frac{1}{2}$

1882

1 I 33, 2 I 43, 3 II 14, 4. III 31

5 The square on any str line, drawn from the vertex of an isosceles Δ to the base, is *less* than the square on a side of the Δ by the rectangle contained by the segments of the base.

6 In a ◯, the extremities of two radii at right ∠s to each other are joined Prove that the ∠ in the segment so formed, is = to *one right angle and a half*.

1883

1 I 41, 2 II 12, 3. III 31, 4 IV 11

5. AB is the hypotenuse of a right angled ΔABC, find a point D in AB such that BD may be = to the ⊥ from D on AC

6 Two tangents are drawn to a ◯ at the opposite extremities of a diameter and cut off from the third tangent a portion AB, if C be the centre of the ◯, show that ∠ACB is a *right* ∠.

1884

1 I 48, 2 II 11, 3 III 22, 4 III 31, 5 IV 10

6 Prove that a parallelogram of which one ∠ is a right ∠, and two adjacent sides are equal, is a *square*.

Show also that its diagonals bisect each other at right ∠s.

7 The str. line drawn from the right ∠ to the middle point of the hypotenuse, is ½ the hypotenuse.

8 Show that in a *regular polygon* of *twelve* sides, each ∠ is 5 times the ∠ subtended by one of the sides at the centre of the circumscribed ⊙

1885.

1 I. 29

Prove that two str. lines bisecting two *adjacent* ∠s of a parallelogram, intersect at right ∠s

2 I 24

3 Given *three* middle points of the sides of a Δ, construct the Δ.

4 II. 7.

5 Two ⊙s touch each other Shew that a str. line drawn through their point of contact, cut off *similar segments* of ⊙s

6 III. 22

7 IV 4

4 Describe a ⊙ touching *three* given str. lines, two of which are parallel

1886

1. I 32 Cor 2, 2 I 43; 3. II 9

4. P and D are any two points, A, B are any two points on opposite sides of the str. line PD. The ΔAPD is = to the ΔBPD, and PD produced cuts AB in C, prove that AC is = to CB

1886-87

1 I 32 Cor 2 2. Shew that an ∠ of a regular *pentagon* is to the ∠ of a regular *decagon* as 3 4. 3. I 43.

4 P and D are any two points, and A and B are any other two points on opposite sides of the str. line PQ. The ΔAPQ = ΔBPQ, and PQ or PQ produced cuts AB in C; prove that AC = CB.

5 II 9 6 If two chords in a ◯, cut each other at right ∠s, the sum of the squares on their segments is = to the square on the diameter

7 III 17 8 A quadrilateral circumscribing a ◯ has two of its sides parallel, show each of the other two sides subtends a right ∠ at the centre

9 Prove that the lines bisecting the ∠s of a regular *pentagon* meet at a point 10 IV 2.

1887-88

1 Ex 2, p 96 Text, 2 I 45, 3 II 12

4 ABC is an equilateral ∠, in BC produced, take any point D, so that BD DC = BC^2, prove that $AD^2 = 2AC^2$

5 III 17, 6 III 37

7 From an external pt O, OP is drawn to touch a ◯, and OQR to cut it, and it is found that OP = 2ce the *radius* and that OR = 2OQ, prove that QR subtends a rt ∠ at the centre

8 IV 3

9 If a ⊙ be inscribed in a right ∠d Δ, the excess of the sides containing the right ∠, over the hypotenuse, is = the diameter of the ◯

1889

1 If one acute ∠ at the base of a Δ, be *double* the other ∠ at the base, and a ⊥ be drawn from the vertex upon the base, shew that the difference between the segments of the base is = to the smaller side

2 In a given point, in a given str line, to make a rectilineal figure = to a given rectilineal figure

3 A is a given point, B is a given point in a given str. line It is required to draw from the given str line, a str line AP such that AP + PB may be = to the *given length*, *greater* than the distance from A to B

4 I 47

5 If any point P be joined to A, B, C, D, the angular points of a rectangle ABCD, then shall $PA^2 + PC^2$ be = $PB^2 + PD^2$, ∠s A and B being opposite to each other

6 II 7

7 In a right ∠d Δ, if a ⊥ be drawn from the vertex to the hypotenuse, the square on the ⊥ is = to the rectangle contained by the segments of the hypotenuse

8 III 32

9 IV 2

1890-91

1 I 21 (*a*) The str lines drawn from any point within a Δ, to its angular points, are together less than the sum of the sides of the Δ

2 ABCD is a rectangle; of which $\angle A$, $\angle C$ are opposite $\angle$s, E is any point in BC, F in CD, shew that $2\Delta AEF+$ rect. BE DF = rect AB CD

3 I 45, 4 II 9

5 If two chords of a ○ intersect at right $\angle$s, the sum of the squares on the 4 segments of the chords = square on the diameter of the ○

6 (*a*) III 33 (*b*) Construct a Δ, having given the base, the vertical $\angle$, and the point in the base on which the $\perp$ falls from the vertical $\angle$.

7. IV 5

8 From the angular pts A, B, C of a Δ, $\perp$s are drawn on the opposite sides, and terminated at the points D, E, F, on the ⊙ce of the circumscribing ⊙, if L be the point of intersection of the $\perp$s, shew that LD, LE and LF are bisected by the sides of the Δ

1891-92

1 (*a*) I 14 (*b*) ABCD is a rhombus, AC is bisected at O If O be joined to the angular points B and D, shew that OB, OD are in *one* str line

2 I 41, 3 If two equal str lines intersect each other anywhere at right $\angle$s, the quadrilateral formed by joining their extremities $=\frac{1}{2}$ the square on either str line

4 II 11, 5 If a str line AB, be divided into any two parts in the point H, such that AB BH $=AH^2$, shew that $AB^2+BH^2=3AH^2$.

6 III 27

7 CD is a chord of a ○ at right $\angle$s to the diameter AB, E is any point in the arc BC; AE cuts CD in F, prove that $\angle DFE=\angle ACE$.

8 Shew that the *perimeter* of a right $\angle$d Δ, exceeds the diameter of the inscribed ○, by *twice* the hypotenuse

9 IV 10

1892-93.

1 I 17 State the *axiom* on which Euclid founds his "*Theory of parallels* Shew that I 17 is the *converse* of Ax. 12

2 I 48

3 The sides AB AC of a Δ, are bisected in F and E, and the lines BE CF are drawn and produced to M and N, so that MB 2BE, and CN = 2CF, prove that MA and AN are in one str line

4 II 13

5 If from one of the extremities of the base of an isosceles Δ, a ⊥ be drawn to the opposite side, then *twice* the rectangle contained by that side, and the segment adjacent to the base, is the square on the base

6 III 3 7 III 22.

8 The str lines which bisect any ∠ of a quadrilateral figure inscribed in a ○ and the opposite exterior ∠, meet on the ○ce

6. Inscribe a *Hexagon* in a ○

9 Two men desire to dig a *well* equidistant from each of two *intersecting straight roads*, and also equidistant from their two *houses* on one of the roads Shew how to find the *position* of the *well*

PUNJAB UNIVERSITY PAPERS.

ENTRANCE EXAMINATION

Geometry.

1875

1 Demonstrate by means of a figure, that under certain conditions I 22 *cannot be solved*

2 I 46, 3 II 14

4 (*a*) What is the difference between "the ∠ *of* a segment," and "the ∠ *in* a segment, of a ○?

(*b*) Prove that the *centre* is the only point in a ○, at which two chords can bisect each other

6 IV 11

7 If in any $\triangle$, the str. line which bisects the vertical $\angle$, also bisects the base, the $\triangle$ is isosceles

8 ABC is a $\triangle$; right-angled at A, and AD is drawn $\perp$ to the base BC; shew that $AB^2 = BC.DB$

9 Divide a given str line into two parts, so that the rectangle contained by its segments, may be = to a given square.

1876

1 I 32, 2. I 43; 3. III 11 and 12, 4 III. 32, 5. III. 35, 6. IV. 10

7. Shew how to *trisect* a given str. line

8 Prove that the difference between two of the sides of any $\triangle$, is *less* than the third side. (See Notes on I. 20)

9 Shew that the difference of the base $\angle$s of any $\triangle$ is *double* the $\angle$ contained by a line drawn from the vertex, $\perp$ to the base, and another bisecting the $\angle$ at the vertex

10 The $\perp$s let fall from the three $\angle$s of any $\triangle$, upon the opposite sides, intersect each other in the same point. (Text p. 224, Ex. 19)

1877

1. (*a*) Define a right $\angle$, a rectangle, a rhombus, a rhomboid, and a trapezium

(*b*) Prove that the diagonals of a *rectangle* are equal, and that those of a *rhombus* cut each other at right $\angle$s.

2 I. 32 Cor 2; 3 II 9

State and prove II. 9 *algebraically*.

4 III. 20; 5 III 22

6 In a given str line, find a point equidistant from two other points, without the line

7 III 33.

8. Prove that the two sides of a $\angle$ are together *greater* than *double* the str line drawn from the vertex of the $\triangle$, to the middle point of the base

9 IV. 15.

10. Draw through a given point, a str. line, which shall make equal $\angle$s with two other given str lines.

1878

1. Define a ⊙ an acute angled △, and parallel str lines

2 (*a*) I 37

(*b*) If P be a point in a side AB of a parallelogram ABCD, and PC, PD be joined, the △PAD + △PBC = PDC

3 Prove II 9 and II 10 together

4 III 12, 5 III 34

6 Given the base of a △, the vertical ∠ and the length of the line drawn from the vertex to the middle point of the base, describe the △

7 IV 6

8 III 9

9 Prove that the point of intersection of the diagonals of a square, described on the hypotenuse of a right ∠d △, is equidistant from the two sides containing the right ∠

1879

1. (*a*) Define —Plane superficies, isosceles △, parallel lines.

(*b*) Mention the *three* Postulates

2 (*a*) I 20

(*b*) If the two diagonals of a four-sided figure, bisect one another, the figure is a parallelogram

3 I 38

4 If two sides of a △ be bisected at right ∠s, and from the point where the bisecting line cut one another, a ⊥ be drawn to the third side, it will bisect the third side

5 II 4, 6 III. 22

7 If in a ⊙, a str line bisect two parallel chords, prove that it passes through the centre

6 III 21

7. IV 10

8 In a given ⊙, to inscribe four equal ⊙s, touching each other and the given ⊙

1881

1 Enunciate and prove Euc I 26

2 Define —Surface, Diameter, Diagonal, Parallel lines, Tangent to a ⊙

3 I. 47.

4. Find the *locus* of a point, from which tangents drawn to two ◯s, are equal, (*i*) when the ◯s touch each other *externally*, (*ii*) when they do not.

5 Inscribe an equilateral Δ in a ◯, and compare its *area* with that of a *regular hexagon* inscribed in the same ◯

6 Two ◯s touch *internally* at A A str line touches one ◯ at P, and cuts the other at *Q* and R Prove that PQ and PR subtend *equal* ∠s at A

7 Enunciate and prove Euc. II 13, and deduce an expression for the *area of the triangle* in terms of its sides

8 Give a geometrical proof of $(x+a)^2-(x-a)^2=4xa$. See II. 8.

1883

1 Classify Δs according to their sides, and draw figures for a *scalene right angled* Δ, an isosceles *obtuse angled* Δ, *rectangle*

2 I 5, case 2nd

3 If AB, AC be equal sides of an isosceles Δ, and a ⊙ with centre B and distance BA, cut AC (or AC produced) in E, and BF be taken in AB (or AB produced, if E lies in AC produced) =to CE, prove that ∠CFA is =to ∠FAC

4 II 5 Cor, 5 III 21

6 If two str lines be drawn through any point on a diagonal of a square parallel to the sides of the square, the points where these st lines meet the sides, will be on the ◯ce of a ◯, whose centre is at the intersection of the diagonals 7. IV 6

8. Having given one side of a Δ, the centre of the circumscribed ◯, determine the *locus* of the centre of the inscribed ◯

9 Draw str lines through the angular points of a parallelogram, which shall form another parallelogram = twice the former

1884

1 Under what circumstances, can you assert the *equality* of two Δs, in every respect?

2. I. 35, 3 II. 14, also I 47, Cor.

4. Prove *geometrically*, that $(a+b)^2+(a-b)^2=2a^2+2b^2$

5 III 17

6 Two ◯s touch internally at A At a point P on the *inner* ◯, a tangent PQR is drawn cutting the other ◯ in Q and R Prove that PQ and PR subtend equal ∠s at A (Cf q 6 of 1881).

7 Define a regular polygon What regular polygons can be inscribed in a ◯, by Euclid's method? Show that, if any regular polygon be *inscribed* in a ◯, a regular polygon of the *same number of sides*, can also be *described about* the same ⊙.

1885

1 Define a *given* st line, a right ∠, parallel st lines, a gnomon, an ∠ *in* a segment of a ◯, and an acute angled triangle.

2 (*a*) I 32

(*b*) ABC is a △, and the exterior ∠s at B and C are bisected by BD and CD respectively, meeting in D Shew that ∠BDC + *half* ∠BAC is = one right ∠

3 II 7, 4 (*a*) III 27

(*b*) A str line is drawn touching a ◯, and parallel to a chord, shew that the *intercepted arc* is bisected at the point of contact

5 Modify the construction of Euc IV 4, so that the ◯ may touch one side of a △ and the other two sides produced Prove the modified proposition (See Text, p 255)

6 IV 9

7 Shew that a ◯ cannot be described about a *rhombus*

1886

1 (*a*) I. 35

(*b*) Bisect a given △, by a str line drawn from any point in a side (Text p 113, Ex 36)

2 ABC is an isosceles △, AB, AC being equal, P is any point in BC, and ⊥s PQ, PR are drawn on the opposite sides Shew that PQ+PR is = the ⊥ from B or C on the opposite side 3 (*a*) II 5

(*b*) Produce a given st line, so that, the rectangle contained by the whole st line thus produced, and the part of it produced, shall be = to a *given square* 4 (*a*) III 20

(*b*) Prove III 22, as a *deduction* from III 20, when the ∠ at the centre, is *greater than*, *equal to*, or *less than* two right ∠s

5 III 26, 6 IV 5

7 A ◯ is described about the △ABC, and the tangent at C meets AB produced at D, prove that the ◯ whose centre is D, and radius DC, cuts AB at E

1887

1. (*a*) What are the different parts of a proposition, in plane Geometry? (*b*) What is the *Converse* of a proposition? (*c*) Is it

necessarily true? (*a*) State the converse of "Any two ∠s of a Δ are together less than two right ∠s." [Ans to (*d*)—Ax. 12]

(*e*) Where is the Converse Prop, first used by Euclid

2 I 32 Cor 2 Is this true for *hexagon*?

3 I. 45, 4. (*a*) II 11

(*b*) In the diagram of II 11, point out any other str lines besides the *given str line*, similarly divided

5. III 34, 6 (*a*) IV 3

(*b*) From the diagram of IV 3, shew that it is possible to describe a Δ equiangular to a given Δ and such that one of its sides and the other two sides produced, shall touch a given ○

7 The str lines drawn from the ∠of a Δ, to the points of bisection of the opposite sides, meet at the same point (Text p 105, Ex. 4.)

8. DR is a diameter of a ⊙; DP, DQ are two chords, meeting the tangent at R, at S and at T respectively. Shew that ∠TPS = ∠TQS

1888

1 (*a*) What supposition is tacitly assumed concerning the "two str lines," in the ordinary enunciation of I 27?

(*b*) Why is (I 27) not necessarily true as it stands? (*d*) After completing the enunciation, prove I. 27

2 I 24 3 Enunciate and prove II 12, 4 III 26.

5 III 11 and 12 What different cases are there of the above

6 (*a*) IV. 10 Give the figure and construction only.

(*b*) In the figure of IV 10, shew that the *smaller* ⊙ is = to the ⊙ described about the *required* Δ 7 I 47 Cor

8 Construct a Δ having given an ∠, and the *radii* of the inscribed and circumscribed ⊙s

1889

1 The area of a parallelogram, is = to the product of the base and the ⊥r height

2 One acute ∠ of a right ∠d Δ, is *double* the other, show that the side opposite to the less, is = to *half* the hypotenuse

3 (*a*) II 4

(*b*) Divide a str line into two parts, such that the rectangle contained by the parts, may be the *greatest* possible.

4 (II 13.) Hence deduce an *expression* for the *area of a triangle, in terms of its sides*, and show what form this expression assumes, when the Δ is right ∠d

5 (*a*) What relation subsists between an ∠ *in* a segment of a ◯, and an ∠ *of* the same segment. (*b*) Divide a ◯ into two segments, so that the ∠ in one segment, may be *double* that in the other.

6 Inscribe an *equilateral* Δ, in a given ◯. (Cf IV 2)

1890

1 I 35, 2 II 5 Cor, 3 III 21, 4 III 29, 5 IV 14

6 Of all Δs on the same base and between the same parallels, the *perimeter* of the isosceles Δ, is the *least*

7. In any ΔABC, if E, F be points where the ⊥s from the opposite ∠s meet the sides AC, AB, prove that $BC^2 = AB \cdot BF + AC \cdot CE$

8 **Simson's line** (See p 232; Text, Ex 74)

1891

1 (*a*) Define a parallelogram, (*b*) I 34, 2 II 14

3 In a given ◯, draw a chord, which will subtend an ∠, at the ◯ce = an ∠ of a regular *hexagon*

4 IV 10, *Construction* only 5 Describe a ◯ about an *obtuse* ∠d Δ (A case of IV 5)

6 A quadrilateral is bounded by (1) the diameter of a ◯, (2) the tangents at the extremities, and (3) a third tangent, shew that its *area* = ½ that of the rectangle contained by the diameter, and the side opposite to it

7 Two diameters AOB, COD of a ◯, are at right ∠s to each other, P is a point in the ◯ce, the tangent at P, meets COD at Q, and AP, BP meet the same st line at R, S respectively Shew that RQ = SQ

8 AB is a fixed chord of a ◯, AC is a *moveable* chord of the same ◯, (∠CAB being therefore *variable*), a parallelogram is described, of which AB and AC are adjacent sides Determine the *greatest* possible *length of the diagonal* through A

1892.

1 (*a*) Define an equilateral Δ, (*b*) I 5 *Cor* (*c*) Enunciate I 6, 2 I 32, 3 The str lines bisecting the ∠s at the base of an isosceles Δ, meet the sides at D and E Shew that DE is parallel to the base

4 I. 38; 5 Mention all the cases in Euc. Bk. I wherein is established the *equality* of two Δs *in area.* 6 II. 12

7. III 20 8 Two tangents AB, AC are drawn to a ◯, D is any point on the ◯ce *outside* the ΔABC, shew that $\angle$ABD + $\angle$ACD is *constant*

9 IV 3, 10 From the 3 $\angle$s of a Δ, $\perp$s are drawn to the opposite sides, meeting them at D, E, F; shew that DE and DF are *equally inclined* to AD

1893

1. I 24, 2 I 46, 3 ABC is an isosceles Δ, the str. line AD bisecting BC is produced to E, and DE made = AD, E is joined to the mid points of AB and AC by str lines cutting BC in F and G. Shew that AFEG is a *rhombus*

(*b*) 4 II 6, 5 (*a*) III 14 (*b*) When can two chords in a ◯, bisect each other

6 IV 7, 7 Divide the ◯ce of a ◯, into *ten* equal parts

8 If two Δs ABC, DEF be inscribed in the same ◯, so that AD, BE, CF intersect each other in one point O, prove that, if O be the centre of the *inscribed* ◯ of one of the Δs, it will be the point of intersection of the $\perp$s in the other, drawn from the angular points on the sides.

1895.

1. I 24, 2. Construct a right $\angle$d Δ, one of the sides containing the rt $\angle$, and the $\perp$ from the right $\angle$, on the hypotenuse being known In which case, will the construction fail

3 (*a*) What do you mean by rectangle contained by two str lines (*b*) II 5

5 III 22, 5 If each pair of opposite sides of a quadrilateral inscribed in a ◯, meet each other when produced, the str lines bisecting the $\angle$s made between them, are at rt $\angle$s to each other

6 (*a*) If two ◯s cut each other, and from any point in the common chord produced, tangents are drawn, one to each ◯; prove that these tangents are equal

(*b*) What are radical axis?

7. Describe a regular *hexagon about* a ◯.

ALLAHABAD ENTRANCE PAPERS

1889

1 Enunciate all the Props of Euc. Bk I. in which the *equality* of three parts in a pair of Δs, involves equality in all respects

2. Construct a Δ, having given the *base*, *one* of the $\angle$s at the *base*, and the *sum of the sides* 3 I 42.

4 From a given point in one of the sides of a Δ, draw a str. line to meet the other side produced so that the Δ thus formed, shall be=to the given Δ

5 II 12, 6 Ex 245; 7 (*a*) III. 34

(*b*) Given the *base*, and *vertical* $\angle$ of a Δ, shew that the Δ is *greatest*, when it is *isosceles*

8 IV 11.

1890

1 Define :—A str line, an acute $\angle$d Δ, a $\odot$, parallel str. lines a gnomon, an $\angle$ *in* a segment, *height* of a Δ When are *magnitudes* said to be equal?

2 I 8 (*Direct proof*), 3 I 29 Case 1

4 If two opposite $\angle$s of a parallelogram be bisected; and two str lines be drawn from the points of bisection to the opposite $\angle$s: these two str lines *trisect* the diagonal

5 I 48 6. Deduce a proof of I 47, from II 4 (See General notes on II. 4)

7 II. 7, 8 III 20, 9. III 35, 10 (*a*) IV. 10 (constr)

10 (*b*) Divide a right $\angle$ into 5 equal parts

1892.

1 Define :—Axiom, postulate, parallel str. lines isosceles Δ, a gnomon, an $\angle$ *in* a segment

2 (*a*) I 20

(*b*) If any point be taken *inside* a quadrilateral, prove that the sum of the distances of the point from the angular points of the quadrilateral, is the *least* possible, when the point is the intersection of the diagonals.

3 II 14, 4. III. 21, 5 (*a*) IV 9

(*b*) If a square be described about the $\bigcirc$ in question 5 (*a*); prove that it is *double* of the *original* square

6 State the enunciations of propositions which relate to the *equality* of two Δs *in all respects*

1893

1 Define —A parallelogram, a secant, a sector, a segment of a ◯, *converse* proposition.

2 What is the difference between "five feet square," and "five square feet"?

3 I 7, 4 I 24; 5 Ex 2 page 96, Text, 6 II 5, Cor

7 If three points are not in the same str. line, a ⊙ may be described, whose ◯ce shall pass through them 8 IV 10

1894.

2. Define —A parallelogram, a gnomon, an arc, a segment of a ◯

2 I 8, 3 I 44, 4 II 7, 5 III 15, 6, IV 5

7 Inscribe a square in a ◯

8 In a ◯, two chords AEA and CED intersect at E, prove that the $\angle$s subtended by AC and BD at the centre, are together $= 2 \angle AEC$

9 Prove the *formula* for determining the *radius* of the ⊙ inscribed in a Δ, whose sides are given

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LONDON MATRICULATION EXAMINATION PAPERS

1897, January.

1 (*a*) Euc I 12

(*b*) Prove that all points in a plane which are equidistant from two given points in the plane, bisect on a str. line

2 (*a*) if AB, CD are two finite str lines, which bisect one another, prove that A, B, C, D are the corners of a ▱m

(*b*) Under what circumstance, is this ▱m (i) a rhombus, (ii) a rectangle?

3 (*a*) Euc I 35 and I 37

(*b*) Find (when possible), a pt P on the ⊙ce of a given ⊙, such that ΔPAB is (of *area*) = to that of the given Δ,—A, B being two given pts in the plane of the ◯

Discuss the number of solutions of this problem

5 Euc II 14, 6 (*a*) Ex. 245 or Ex. 24, p 147, Text

(*b*) A pt is such that the sum of the sq on the str lines joining it, to 4 fixed pts is given, the fig. being in one plane, prove that the pt lies on a fixed ◯

7 Euc III 20

8 (*a*) If two ◯s intersect one another, prove that the tangents drawn to them, from any pt on their common chord produced — are equal to one another

(*b*) Show how to find pt, the tangents from which, have a similar property, when the two ◯s do not intersect one another

8 Construct a common tangent to two ◯s

(See Ex. Bk III, Text)

How many such tangents can be drawn? Draw the figure to illustrate the different cases

10 (*a*) Euc IV 10

(*b*) Hence, show how to inscribe in a ◯—a *regular polygon* of 10 sides

1890

1 Euclid I 16

2 " I 35

3 " II 9

4. If O be any point on the base AC of the isosceles △ABC, prove that the rectangle contained by AO and OC, is = to the differerence of the squares on AB and OB

5 If CD be any chord of a ◯, P any point on a diameter parallel to CD, and QD the point on the ◯ which is farthest away from the chord CD Prove that the square on PC and the square on PD, are together double the square on PQ

6 III 37

7 Prove that a ◯ can be circumscribed about any quadrilateral, whose sides are the bisectors of the ∠s of some other quadrilateral

8 If P, Q be the points of intersection of two ◯s, if a str line cut one of the ◯s at A and D, and the other at B and C Prove that the ∠APB is = to the ∠CQD [In drawing the figure, after making the points A and D, place C on the same side of B as D is of A.]

9 Describe a circle through three given points

10 Having given the base, and the vertical ∠ of a △, prove that the bisector of the vertical ∠ always passes through one of the two fixed points

1891, January

1. Prove that the diagonals of a parallelogram, bisect each other.

2. Squares are described on the three sides of a right ∠d Δ; divide the square on the hypotenuse into two rectangles, which shall be respectively = to the squares on the other sides (Give the proof) 3 (*a*) I. 22. When is the construction impossible?

4 (*b*) Show that, if the square on one of the lines exceeds the sum of the squares on the other two, the Δ will have an obtuse ∠d.

4 Construct a square which shall be = to a given Δ (II. 14.)

5 Prove that the sum of the squares on the sides of a parallelogram, is = to the sum of the squares on its diagonals

6. III. 33, 7. III. 15

8 (*a*) In a ΔABC, the ⊥ AD, BE, drawn from two vertices to the opposite sides, meet in a point O, and AD meets the ⊙ circumscribed about the Δ in a pt. K. Prove that DK = DO.

8 (*b*) Deduce that the ⊥ drawn from the *third* vertex C to the opposite side, also passes through O

9 (*a*) IV. 11. (*b*) Prove that, if its alternate sides are produced to meet, the ∠s of the *star-shaped* figure thus formed, are each = two-fifths of a right ∠.

10 Show, by aid of a diagram, that four ⊙s can be drawn, so as to touch each of three given str. lines, and give constructions for their centres

1891, June.

1. Erect a ⊥ to a given str. line AB, at the point A (I. 11.) How would you construct the ⊥, when the given str. line terminates at A, and it is not allowed to produce it?

2 If two Δs of equal area, are in the same plane and on the same base, the str. line joining their vertices, is either parallel to the base, or is bisected by the base

3. Give a def. of a square free from redundant conditions. If the ∠A of a ΔABC, be a rt. ∠, and AD be drawn ⊥ to BC, prove that AB^2 is = BC.BD 4. II. 7.

5 Of all rectangles of given perimeter, that which has the *greatest* area, is a square

6. (*a*) If the ∠A of a ΔABC be obtuse, the square on BC exceeds the sum of the squares on BA, AC, by twice a certain rectangle. State what rectangle this is, and prove the theorem (II. 12.)

(*b*) The side QR of an equilateral ΔPQR is produced to S, so that RS=QR Prove that $PS^2=3\ PS^2$

7 (*a*) III 20 (*b*) A, B, are any two points on opposite sides of a str line CD, find at what points of CD, they subtend a rt $\angle$.

8 AB, CD are two str lines, which, being produced, intersect in O, prove that if OA OB=OC CD, the four points A, B, C, D, lie on a ◯

9 Describe a ◯ touching a given str line at a given point, and passing through another given point

10 Given a square, show how to obtain from it, a regular *octagon*, by cutting off the corners

1892, January

1 I 24

2 Prove that if D and E be the middle points of the sides AB, AC of a Δ, then DE is parallel to BC If the median BE= median ED, prove that the Δ is isosceles

3 Construct a square on a given str line AB as a diagonal

Prove that, if ABCD be the square, and if the bisector of the $\angle$BAC meet BC in E, then CE is=AC$\sim$AB

4 Prove geometrically, that the square on any two str lines exceeds twice the rectangle contained by the str lines by the square on the difference of the str lines 5 II 11

6 If A, B C be any three points, and O the middle point of BC, then AC^2+AB^2=double of the squares on AO and OC

7 (*a*) III 15 (*b*) The *greatest* rectangle which can be inscribed in a given ◯, is a square

8 III 22

9 Describe a ◯ passing through two given points, and having its centre on a given str line.

10 (*a*) Prove that, if the circumference of a ◯ be divided into any number of equal arcs, the polygon formed by the chords of these arcs, has all its sides equal, and all its $\angle$s equal

(*b*) Also, prove that the ◯ through the centre and extremities of any side, touches the adjacent sides, and that the radii of the first ◯ drawn to the vertices, cut the second ◯ in points which form the vertices of another regular polygon

to the sides, such that a pair of opposite parallelograms formed by them, are equal in area, prove that the point must lie on a diagonal of the original parallelogram

4. (*a*) Euc II 4 (*b*) Also, inscribe in the above larger square, a rectangle contained by the diagonals of the smaller squares and show that it is=twice the rectangle contained by the segments of the str line

5 Construct a rectangle=a given square, and such that the sum of its sides, shall be=to a given str line

6 Prove that the shortest chord of a ○, which can be drawn through a given point inside it, is at rt ∠s to the str line joining that point to the centre

7 (*a*) III 21. (*b*) Deduce that a Δ with given base and given vertical ∠ is of greatest area, when it is isosceles

8 Three points A, B, C are taken on a ○ and a str. line parallel to the tangent at A, intersects the str lines AB, AC, in the points D, E, prove that a ○ may be described through the four points B, C, D, E

9 (*a*) Euc. IV 5 What is the condition that the centre of the ○ should be outside the Δ

(*b*) For a ΔABC, find a point D on the base BC produced, such that AD² is=DB DC

10 In a regular *heptagon*, the extremities of a side DE are joined to the opposite vertex A; prove that in the ΔADE, each ∠ at the base is=three times the vertical ∠

1893, January

1. (*a*) Euc I 20

(*b*) Prove that if a ΔDEF lies wholly inside another ΔABC, it has a smaller perimeter

2 If in two Δs FGH, LMN, the sides FG, FH are respectively=to the sides LM, LN, and the ∠FGH is=to the ∠LMN Shew that the ∠s GHF, MNL are either equal, or are together=to two rt. ∠s

3 Two str lines AOB, COD are such, that the Δs AOC, BOD are=in area Prove that AD is parallel to BC.

4 The alternate sides of *pentagon* are produced to meet, so as to form a 'star-shaped' figure Prove that the sum of the internal ∠s at the vertices of the *star*, is=to two rt ∠s Find also the value of the corresponding sum, for the general case of a polygon of *n* sides.

5. Four points A, B, C, D are taken in this order, in a str. line Shew that the sum of the squares on AD, BC exceeds the sum of the squares on AC, BD, by twice the rectangle contained by AB,CD (A proof by geometrical construction, will be preferred)

6 Euc II 13

7 Two str bars AX, BY rotate round the fixed points A, B with equal speed, that is, so that the ∠s LAX, MBY are always equal, L and M being fixed points Show that when they rotate in the same direction, their point of intersection describes a ◯

8 From a point M in a chord AB of a ◯, whose centre is O, a str line is drawn at rt ∠s to OM, and meeting the ◯ in N, prove that AM MB is $=MN^2$

Show how to draw a str line, such the square on it, is = to the given rectangle, when the point M is on the chord produced

9 Prove that, if a ◯ can be drawn to touch all four sides of a quadrilateral, the sum of one pair of opposite sides, must be = to the sum of the other pair

10 Prove that, the str lines bisecting the ∠s of a Δ, meet in a common point, show that this point is nearest to the greatest ∠ of the Δ and furthest from the least ∠

1893, June

1 Euc I 8, 2 (*a*) Euc I 32

2 (*b*) If ABC be a rt ∠d Δ, and AO be drawn making the ∠ BAO = to ∠ABC, and meeting the hypotenuse BC in O, then O bisects BC Also shew that, if OH be drawn ⊥ to BC, on the side remote from A and = to OA, then HA bisects the rt ∠ BAC

3 If two Δs have two sides of the one, respectively = to two sides of the other, and if the ∠ opposite one pair of equal sides be rt ∠s, the Δs are = in all respects

4 State and prove a geometrical proposition, answering to the algebraical identity, $(a-b)^2+2ab=a^2+b^2$

Also, explain (without proof), what geometrical proposition corresponds to the identity $\left(\frac{a+b}{2}\right)^2-\left(\frac{a+b}{2}\right)^2=ab$

5 Euc. II 11

6 Make a square = to given rectangle Also, construct a rectangle of given perimeter, which shall be = a given square. When is the problem impossible?

7 Euc. III 20, & III. 33.

9 Prove that, one and only one ○ can be drawn through three given points, which are not in the same str. line

10. Draw three ○s of given (unequal) radii, each of which touches the other two, the points of contact, being distinct In what case, has the problem more than one solution?

January, 1894

1 Euc I 16

2 Prove, by Euclid Book I that the str line joining the middle points of two sides of a Δ, is parallel to the third side

If D be the middle point of the side BC of a ΔABC, and E the middle point of AD, and if BE produced meet the side AC in F, then AF is = one-third of AC

3 (*a*) Euc I 47

(*b*) Prove also, by means of the same figure, that the square on AC is less than the squares on AB and BC, by twice the rect BC BD, where D is the foot of the ⊥ from from A on BC

5 (*a*) Euc II. 14.

(*b*) Prove that the diagonal of the square, will be less than that of the rectangle

6 Of two chords of a ○, that which subtends the greater ∠ at the centre, is the greater

7 Euc III 30

8 Describe a ○ touching a given str line at a given point, and passing through a given point external to the str. line

9 Describe a Δ, whose sides should touch a given ○, and whose ∠ shall be respectively = to three given ∠s, whose sum is two rt ∠s

10 Euc IV 15

1894, June.

1 (*a*) Euc I 8.

(*b*) Two equal str lines AB and CD are joined towards opposite parts by the equal str lines AD and BC intersecting in O Prove that the Δs OAC and OBD are isosceles

2. Euc I 21

3 (*a*) Euc I 34

(*b*) Upon the same base AB, and upon opposite sides of it, the parallelograms ABCD and ABEF are described, so that the side

AD of the first, is = to the diagonal AE of the second, and the diagonal AC of the first, is = to the side AF of the second Prove that AC and AF, as also AD and AE, are in the same str. line

4 (*a*) I 39

(*b*) The point P is given in position within the ∠AOB between two infinite str lines OA and OB, and the str line QPR is drawn through A, cutting OA and OB in Q and R Prove that if the area of the △QOR be the least possible, it will be bisected by OP

5 Euc I 41, 6 II 11, 7 Euc. II 14

8 (*a*) Euc. II 22

(*b*) From the point T, external to the circle QPR, whose centre is O, str lines TP and TRQ are drawn touching the ⊙ in P, and cutting it in Q and R respectively, and QR is bisected in S, prove that (i) ∠ PST = ∠POT

(ii) If PS produced meets the ○ again in U, and UV drawn parallel to QR meets the ○ again in V, then TV will touch the ○

6 Euc. III. 36

10 (*a*) Euc IV 4

(*b*) If a number of △s have one side and the centre of the circumscribing ○ common to all, prove that the centres of the inscribed ○ will all lie on the circumference of a certain fixed ○

1895, January

1 If the opposite sides of a four-sided figure be equal, the opposite ∠s shall also be equal

2 (*a*) Euc I 12

(*b*) Also prove that only one such ⊥ can be drawn

3 If P and P be two points on one side of a str line AB, such that the ⊥s from them to AB, are equal, then PQ is parallel to AB

4 Euc I 48

5 Prove geometrically, that the sum of the squares on any two str lines cannot be less than twice the rectangle contained by those str lines

6 Euc II 11

7 (*a*) Euc III. 21

(b) Two equal ⊙s intersect in A and B, and any str line through A, meets the ⊙s again in P and Q Prove that P and Q are equidistant from B

8 Euc III. 17, Case (2)

(b) Two tangents are drawn at the extremities of a diameter AB of a ⊙, and a third tangent meets them in P and Q Prove that rect AP.BQ is=to the square on the radius of the ⊙

9 Euc III 35, Case (3)

10 IV 5

In what cases will the centre be (i) *inside*, and (ii) *outside* the Δ

1895, June

1 (a) Euc I 18

(b) No str line can be drawn within a Δ, greater than the greatest side.

2 (a) Euc I 37

(b) The opposite sides AD,BC of the quadrilateral ABCD are parallel, E is the intersection of the diagonals, and F is the middle point of BC, prove that the ΔAEF=the ΔDEF

3 Euc. I. 43, 4 Euc II 7

5 (a) Euc II 14

(b) Of all parallelograms of given *area*, the square has the least perimeter

6 Euc III 2, 7 Euc III 26

8 (a) Euc III 32

(b) Each of four unequal ⊙s, touches externally two and only two of the remaining three, prove that a ⊙ can be drawn through the *four* points of contact 9 IV 7.

10 Divide a right angle into five equal parts

1896, January

1 (a) Euc I 11

(b) Find a point in a given str. line, which is at equal distances from two given points

3 (a) Euc I 32

(b) On the sides BC,CA, AB of an equilateral Δ, three equilateral Δs BCD, CAF, ABF are drawn, prove that D, E, and F are at the corners of an equilateral Δ

3 Describe a rectangle = to a given triangle (I 42)

4 (*a*) Euc I 47

(*b*) Describe a square, which is *three* times a given square

5 (*a*) Euc II 9, 6 Euc III 17

7 (*a*) Euc III 31

(*b*) Construct a rt ∠d △, with the rt. ∠ at a given point, the base of given length and its extremities on two given parallel str lines Under what conditions, is this problem, a possible one?

8 Euc III 35

9 (*a*) Euc IV 4

(*b*) How many such ○s can be drawn Illustrate your answer by a figure?

10 (*a*) IV 15

(*b*) If any hexagon ABCDEF circumscribes a ⊙, prove that the sum of the sides AB, CD, EF is = to the sum of the sides BC, DE, FA

THE END